REGULAR SAMPLING ON METABELIAN NILPOTENT LIE GROUPS

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ABSTRACT. Let N be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$ having rational structure constants. We assume that N = P ⋊ M, M is commutative, and for all $\lambda \in \mathfrak{n}^*$ in general position the subalgebra $\mathfrak{p} = \log(P)$ is a polarization ideal subordinated to λ ($\mathfrak{p}$ is a maximal ideal satisfying $[\mathfrak{p}, \mathfrak{p}] \subseteq \ker \lambda$ for all λ in general position and $\mathfrak{p}$ is necessarily commutative.) Under these assumptions, we prove that there exists a discrete uniform subgroup $\Gamma \subset N$ such that $L^2(N)$ admits band-limited spaces with respect to the group Fourier transform which are sampling spaces with respect to Γ. We also provide explicit sufficient conditions which are easily checked for the existence of sampling spaces. Sufficient conditions for sampling spaces which enjoy the interpolation property are also given. Our result bears a striking resemblance with the well-known Whittaker-Kotel'nikov-Shannon sampling theorem.

1. INTRODUCTION

It is a well-established fact that a band-limited function on the real line with its Fourier transform vanishing outside of an interval $\left[-\frac{\Omega}{2}, \frac{\Omega}{2}\right]$ can be reconstructed by the Whittaker-Kotel'nikov-Shannon sampling series from its values at points in the lattice $\frac{1}{\Omega} \mathbb{Z}$ (see [14]). This series expansion takes the form

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{\Omega}\right) \frac{\sin\left(\pi\Omega\left(t - \frac{k}{\Omega}\right)\right)}{\pi\Omega\left(t - \frac{k}{\Omega}\right)}$$

with convergence in $L^2(\mathbb{R})$ as well as convergence in $L^\infty(\mathbb{R})$. A relatively novel problem in harmonic analysis has been to find analogues of Whittaker-Kotel'nikov-Shannon sampling series for non-commutative groups. Since $\mathbb{R}$ is a commutative nilpotent Lie group, it is natural to investigate if it is possible to extend Whittaker-Kotel'nikov-Shannon's theorem to nilpotent Lie groups which are not commutative.

Let G be a locally compact group and Γ a discrete subset of G. Let **H** be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions. We say that **H** is a **sampling space** with respect to the set Γ [9] if the following conditions are satisfied. Firstly, the restriction map $f \mapsto f|_\Gamma$ defines a constant multiple of an isometry of **H**

into the Hilbert space of square-summable sequences defined over Γ. In other words, there exists a positive constant $c_{\mathbf{H}}$ such that

$$(1) \qquad\qquad \sum_{\gamma \in \Gamma} |f(\gamma)|^2 = c_{\mathbf{H}} \|f\|_2^2$$

for all f in $\mathbf{H}$. Secondly, there exists a vector s in $\mathbf{H}$ such that an arbitrary element f in the given Hilbert space has the expansion

$$(2) \qquad\qquad f(x) = \sum_{\gamma \in \Gamma} f(\gamma) s\left(\gamma^{-1}x\right)$$

with convergence in the norm of $L^2(G)$. If Γ is a discrete subgroup of G, we say that $\mathbf{H}$ is a **regular sampling space** with respect to Γ. Also, if $\mathbf{H}$ is a sampling space with respect to Γ and if the restriction mapping $f \mapsto f|_\Gamma \in l^2(\Gamma)$ is surjective then we say that $\mathbf{H}$ has the **interpolation property** with respect to Γ. This notion of sampling space is taken from [9] and is analogous to Whittaker-Kotel'nikov-Shannon's theorem. In [10], the authors used a less restrictive definition. They defined a sampling space to be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions with the additional requirement that the restriction map $f \mapsto f|_\Gamma$ is a topological embedding of $\mathbf{H}$ into $l^2(\Gamma)$ in the sense that there exist positive real numbers $a \leq b$ such that

$$a\|f\|_2^2 \leq \sum_{\gamma \in \Gamma} |f(\gamma)|^2 \leq b\|f\|_2^2$$

for all $f \in \mathbf{H}$. The positive number b/a is called the tightness of the sampling set. Notice that in (1) the tightness of the sampling is required to be equal to one. Using oscillation estimates, the authors in [10] provide general but precise results on the existence of sampling spaces on locally compact groups. The band-limited vectors in [10] are functions that belong to the range of a spectral projection of a self-adjoint positive definite operator on $L^2(G)$ called the sub-Laplacian. This notion of band-limitation is essentially due to Pesenson [20] and does not rely on the group Fourier transform.

We shall employ in this work a different concept of band-limitation which in our opinion is consistent with the classical one (Whittaker-Kotel'nikov-Shannon band-limitation), and the main objective of the present work is to prove that under reasonable assumptions (see Condition 3) Whittaker-Kotel'nikov-Shannon Theorem naturally extends to a large class of non-commutative nilpotent Lie groups of arbitrary step.

Let G be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. A subspace $\mathbf{H}$ of $L^2(G)$ is said to be a **band-limited space** with respect to the group Fourier transform if there exists a bounded subset E of the unitary dual of the group G such that E has positive Plancherel measure, and $\mathbf{H}$ consists of vectors whose group Fourier transforms are supported on the bounded set E. In this work, we address the following.

Problem 1. *Let* N *be a simply connected and connected nilpotent Lie group with Lie algebra* $\mathfrak{n}$ *with rational structure constants. Are there conditions on the Lie algebra* $\mathfrak{n}$ *under which there exists a uniform discrete subgroup* $\Gamma \subset N = \exp\mathfrak{n}$ *such that* $L^2(N)$

admits a band-limited (in terms of the group Fourier transform) sampling subspace with respect to Γ?

Firstly, we observe that if $N = \mathbb{R}^d$ then Γ can be taken to be an integer lattice, and the Hilbert space of functions vanishing outside the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ is a sampling space which enjoys the interpolation property with respect to $\mathbb{Z}^d$. Secondly, let N be the Heisenberg Lie group and let Γ be a discrete uniform subgroup of N which we realize as follows:

$$N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \text{ and } \Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : k, l, m \in \mathbb{Z} \right\}.$$

It is shown in [9, 6] that there exist subspaces of $L^2(N)$ which are sampling subspaces with respect to Γ. We have also established in [19, 18, 17] the existence of sampling spaces defined over a class of simply connected, connected nilpotent Lie groups which satisfy the following conditions: N is a step-two nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n such that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{c}$ where $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{c}$, $\mathfrak{a}, \mathfrak{b}$ are commutative Lie algebras, $\mathfrak{a} = \mathbb{R}\text{-span}\{X_1, X_2, \cdots, X_d\}$, $\mathfrak{b} = \mathbb{R}\text{-span}\{Y_1, Y_2, \cdots, Y_d\}$, $\mathfrak{c} = \mathbb{R}\text{-span}\{Z_1, Z_2, \cdots, Z_{n-2d}\}$ $(d \geq 1, n > 2d)$ and

$$(3) \qquad (Z_1, \cdots, Z_{n-2d}) \mapsto \det \begin{bmatrix} [X_1, Y_1] & \cdots & [X_1, Y_d] \\ \vdots & \ddots & \vdots \\ [X_d, Y_1] & \cdots & [X_d, Y_d] \end{bmatrix}$$

is a non-vanishing polynomial in the variables $Z_1, \cdots, Z_{n-2d}$. To the best of our knowledge, prior to this work, regular sampling for band-limited (in terms of the group Fourier transform) left-invariant spaces defined over nilpotent Lie groups has only been systematically studied on step one (the classical Euclidean case) and some step two nilpotent Lie groups [6, 9, 19, 17, 18]. We shall prove that for any given natural number k, there exists a nilpotent Lie group of step k which admits band-limited sampling spaces in terms of the Plancherel transform with respect to a discrete uniform subgroup (see Example 6).

1.1. **Overview of the Paper.** Let us start by fixing notation and by recalling some relevant concepts.

- Let Q be a linear operator acting on an n-dimensional real vector space V. The norm of the matrix Q induced by the max-norm of the vector space V is given by

$$\|Q\|_\infty = \sup\left\{\|Qx\|_{\max} : x \in V \text{ and } \|x\|_{\max} = 1\right\}$$

and the max-norm of an arbitrary vector is given by

$$\|x\|_{\max} = \max\left\{|x_k| : 1 \leq k \leq n\right\}.$$

Next, letting $[Q]$ be the matrix representation of Q with respect to a fixed basis, the transpose of this matrix is denoted $[Q]^T$.

- Given a countable sequence $(f_i)_{i \in I}$ of vectors in a Hilbert space $\mathbf{H}$, we say that $(f_i)_{i \in I}$ forms a **frame** [2, 13, 21] if and only if there exist strictly positive real numbers a, b such that for any vector $f \in \mathbf{H}$,

$$a \, \|f\|^2 \le \sum_{i \in I} |\langle f, f_i \rangle|^2 \le b \, \|f\|^2 \, .$$

  In the case where $a = b$, the sequence $(f_i)_{i \in I}$ is called a **tight frame**. If $a = b = 1$, $(f_i)_{i \in I}$ is called a **Parseval frame**.

- Let π be a unitary representation of a locally compact group G acting on a Hilbert space $\mathbf{H}_\pi$. We say that the representation π is **admissible** [9] if there exists a vector h in $\mathbf{H}_\pi$ such that the linear map

(4)
$$g \mapsto V_h^\pi (f) = \langle f, \pi (g) h \rangle$$

  defines an isometry of the Hilbert space $\mathbf{H}_\pi$ into $L^2 (G)$. In this case, the vector h is called an **admissible vector** for the representation π.

- Let $(A, \mathcal{M})$ be a measurable space. A family $(\mathbf{H}_a)_{a \in A}$ of Hilbert spaces indexed by the set A is called a field of Hilbert spaces over A [7]. An element f of $\Pi_{a \in A} \mathbf{H}_a$ is a vector-valued function $a \mapsto f(a) \in \mathbf{H}_a$ defined on the set A. Such a map is called a **vector field** on A. A **measurable field** of Hilbert spaces defined on a measurable set A is a field of Hilbert spaces together with a countable set $(e_j)_{j \in J}$ of vector fields such that the functions $a \mapsto \langle e_j (a), e_k (a) \rangle_{\mathbf{H}_a}$ are measurable for all $j, k \in J$, and the linear span of $\{e_j (a)\}_{j \in J}$ is dense in $\mathbf{H}_a$ for each a. A vector field f is called a **measurable vector field** if $a \mapsto \langle f(a), e_j (a) \rangle_{\mathbf{H}_a}$ is a measurable function for each index j.

- Let $\mathfrak{n}$ be a nilpotent Lie algebra of dimension n, and let $\mathfrak{n}^*$ be the dual vector space of $\mathfrak{n}$. A **polarizing subalgebra** $\mathfrak{p}(\lambda)$ subordinated to a linear functional $\lambda \in \mathfrak{n}^*$ (see [3, 16]) is a maximal algebra satisfying

$$[\mathfrak{p}(\lambda), \mathfrak{p}(\lambda)] = \text{Span-} \{[X, Y] \in \mathfrak{n} : X, Y \in \mathfrak{p}(\lambda)\} \subseteq \ker (\lambda) \, .$$

- The coadjoint action on the dual of $\mathfrak{n}$ is the dual of the adjoint action of $N = \exp \mathfrak{n}$ on $\mathfrak{n}$. In other words, for $X \in \mathfrak{n}$, and a linear functional $\lambda \in \mathfrak{n}^*$, the coadjoint action is defined as follows:

(5)
$$(\exp X \cdot \lambda)(Y) = \left\langle \left(e^{\text{ad-}X}\right)^* \lambda, Y \right\rangle = \left[\left(e^{\text{ad-}X}\right)^* \lambda\right](Y) \, .$$

The following is a concept which is central to our results.

Definition 2. *Let $\mathfrak{p}$ be a subalgebra of $\mathfrak{n}$. We say that $\mathfrak{p}$ is a **constant polarization subalgebra** of $\mathfrak{n}$ if there exists a Zariski open set $\Omega \subset \mathfrak{n}^*$ which is invariant under the coadjoint action of N and $\mathfrak{p}$ is a polarization subalgebra subordinated to every linear functional in Ω.*

In other words, $\mathfrak{p}$ is a constant polarization subalgebra of $\mathfrak{n}$ if $\mathfrak{p}$ is a polarization algebra for all linear functionals in general position, and it can then be shown (see Proposition 7) that $\mathfrak{p}$ is necessarily commutative.

1.1.1. *Summary of Main Results.* Let us suppose that $N = P \rtimes M = \exp(\mathfrak{p}) \rtimes \exp(\mathfrak{m})$ is a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{m}$ such that

Condition 3.

(1) $\mathfrak{p}$ *is a constant polarization ideal of $\mathfrak{n}$ (thus commutative) $\mathfrak{m}$ is commutative as well,* $p = \dim \mathfrak{p}$, $m = \dim \mathfrak{m}$ *and* $p - m > 0$.

(2) *There exists a strong Malcev basis* $\left\{ Z_1, \cdots, Z_p, A_1, \cdots, A_m \right\}$ *for $\mathfrak{n}$ such that* $\left\{ Z_1, \cdots, Z_p \right\}$ *is a basis for $\mathfrak{p}$ and* $\{ A_1, \cdots, A_m \}$ *is a basis for $\mathfrak{m}$ and*

$$\Gamma = \exp\left(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p \right) \exp\left(\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_m \right)$$

is a discrete uniform subgroup of N. *This is equivalent to the fact that $\mathfrak{n}$ has rational structure constants (see Chapter* 5, [3]*).*

In order to properly introduce the concept of band-limitation with respect to the group Fourier transform, we appeal to Kirillov's theory [3] which states that the unitary irreducible representations of N are parametrized by orbits of the coadjoint action of N on the dual of its Lie algebra and can be modeled as acting in $L^2(\mathbb{R}^m)$. Let Σ be a parameterizing set for the unitary dual of N. In other words, Σ is a cross-section for the coadjoint orbits in an N-invariant Zariski open set $\Omega \subset \mathfrak{n}^*$. If the ideal $\mathfrak{p}$ is a constant polarization for $\mathfrak{n}$ then the orbits in general position are 2m-dimensional submanifolds of $\mathfrak{n}^*$ and we shall (this is a slight abuse of notation) regard Σ as a Zariski open subset of $\mathbb{R}^{p-m} = \mathbb{R}^{n-2m}$. Next, let L be the left regular representation of N acting on $L^2(N)$ by left translations. Let

$$\mathcal{P} : L^2(N) \longrightarrow L^2\left(\Sigma, L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m), d\mu(\lambda) \right)$$

be the Plancherel transform which defines a unitary map on $L^2(N)$ (see Subsection 2.2.2). The Plancherel transform intertwines the left regular representation with a direct integral of irreducible representations of N. The measure used in the decomposition is the so-called Plancherel measure: $d\mu$; which is a weighted Lebesgue measure on Σ. More precisely $d\mu(\lambda)$ is equal to $|\mathbf{P}(\lambda)| d\lambda$ where $\mathbf{P}(\lambda)$ is a polynomial defined over Σ and $d\lambda$ is the Lebesgue measure on Σ (see Lemma 23.) Given a μ-measurable bounded set $\mathbf{A} \subset \Sigma$, and a measurable field of unit vectors $(\mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$ in $L^2(\mathbb{R}^m)$, the Hilbert space $\mathbf{H_A}$ which consists of vectors $f \in L^2(N)$ such that

$$\mathcal{P}f(\lambda) = \begin{cases} \mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda) \text{ if } \lambda \in \mathbf{A} \\ 0 \otimes 0 \text{ if } \lambda \notin \mathbf{A} \end{cases}$$

and $(\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$ is a measurable field of rank-one operators is a left-invariant multiplicity-free band-limited subspace of $L^2(N)$ which we identify with $L^2(\mathbf{A} \times \mathbb{R}^m)$. Conjugating the operators L(x) by the Plancherel transform, we obtain that

$$[\mathcal{P} \circ L(x) \circ \mathcal{P}^{-1}](\mathbf{v}_\lambda \otimes \mathbf{u}_\lambda)_{\lambda \in \Sigma} = ([\sigma_\lambda(x)\mathbf{v}_\lambda] \otimes \mathbf{u}_\lambda)_{\lambda \in \Sigma} \equiv \sigma_\lambda(x)\mathbf{v}(\lambda, \cdot)$$

where σ_λ is the unitary irreducible representation corresponding to the linear functional $\lambda \in \Sigma$. Let $L_{\mathbf{H_A}}$ be the representation induced by the action of the left regular

representation on the Hilbert space $\mathbf{H_A}$. It can be shown that if the spectral set $\mathbf{A}$ satisfies precise conditions specified in Theorem 4 then the restriction of $L_{\mathbf{H_A}}$ to the discrete group Γ is unitarily equivalent with a subrepresentation of the left regular representation of Γ acting on $l^2(\Gamma)$. The existence of band-limited sampling spaces with respect to Γ can then be established by directly appealing to known results contained in the Monograph [9]. Define $\beta : \Sigma \times \mathbb{R}^m \to \mathbb{R}^p$ such that

$$\beta(\lambda, t) = \exp(t_1 A_1 + \cdots + t_m A_m) \cdot \lambda|_{\mathfrak{p}^*}$$

where $t = (t_1, \cdots, t_m)$. Under the assumptions listed in Condition 3, it is worth noting that β is a diffeomorphism (Lemma 19) and the following holds true.

Theorem 4. *Let* $N = PM = \exp(\mathfrak{p})\exp(\mathfrak{m})$ *be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ satisfying Condition 3. Let $\mathbf{A}$ be a μ-measurable bounded subset of Σ.*

(1) *If $\beta(\mathbf{A} \times [0,1)^m)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$ then there exists a vector $\eta \in \mathbf{H_A}$ such that $V_\eta^L(\mathbf{H_A})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.*

(2) *If $\beta(\mathbf{A} \times [0,1)^m)$ is equal to a fundamental domain of $\mathbb{Z}^p$ then there exists a vector $\eta \in \mathbf{H_A}$ such that $V_\eta^L(\mathbf{H_A})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with the interpolation property with respect to Γ.*

Let $s = (s_1, s_2, \cdots, s_m)$ be an element of $\mathbb{R}^m$ and define $A(s)$ to be the restriction of the linear map $\text{ad}\left(-\sum_{j=1}^m s_j A_j\right)$ to the ideal $\mathfrak{p} \subset \mathfrak{n}$. Let $[A(s)]$ be the matrix representation of the linear map $A(s)$ with respect to the basis $\{Z_1, \cdots, Z_p\}$. Let $e^{[A(s)]}$ be the matrix obtained by exponentiating $[A(s)]$. Since $s \mapsto \left\|e^{[A(s)]^T}\right\|_\infty$ is a continuous function of s it is bounded over any compact set and in particular over the cube $[0,1]^m$. As such, letting ε be a positive real number satisfying

$$(6) \qquad \varepsilon \le \delta = \frac{1}{2}\left(\sup\left\{\left\|e^{[A(s)]^T}\right\|_\infty : s \in [0,1)^m\right\}\right)^{-1} < \infty,$$

we shall prove that under the assumptions provided in Condition 3, the set

$$B(\varepsilon) = \beta\left((-\varepsilon, \varepsilon)^{n-2m} \times [0,1)^m\right)$$

has positive Lebesgue measure and is contained in a fundamental domain of $\mathbb{Z}^p$. Appealing to Theorem 4, we are then able to establish the following result which provides us with a concrete formula for the bandwidth of various sampling spaces.

Corollary 5. *Let* $N = PM = \exp(\mathfrak{p})\exp(\mathfrak{m})$ *be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ satisfying Condition 3. For any positive number ε satisfying (6) there exists a band-limited vector $\eta = \eta_\varepsilon$ in the Hilbert space $\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}$ such that $V_\eta^L\left(\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.*

Next, we exhibit several examples to illustrate that the class of groups under consideration is fairly large.

Example 6.

(1) *Let* N *be a simply connected, connected nilpotent Lie group with Lie algebra* $\mathfrak{n}$ *of dimension four or less. Then, there exists a uniform discrete subgroup* $\Gamma \subset N$ *such that* $L^2(N)$ *admits a band-limited sampling subspace with respect to* Γ. *Additionally, the Heisenberg Lie group admits a sampling space which has the interpolation property with respect to a uniform discrete subgroup.*

(2) *Let* N *be a simply connected, connected nilpotent Lie group with Lie algebra spanned by* $Z_1, Z_2, \cdots, Z_p, A_1$, *the vector space generated by* $Z_1, Z_2, \cdots, Z_p$ *is a commutative ideal,* $[\text{ad}A_1]|_{\mathfrak{p}} = A$ *is a nonzero rational upper triangular nilpotent matrix of order* p, *and* $e^A \mathbb{Z}^p \subseteq \mathbb{Z}^p$. *Then* $L^2(N)$ *admits a band-limited sampling subspace with respect to the discrete uniform subgroup*

$$\exp\left(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p\right)\exp\left(\mathbb{Z}A_1\right).$$

(3) *Let* N *be a simply connected, connected nilpotent Lie group with Lie algebra spanned by* $Z_1, Z_2, \cdots, Z_p, A_1, \cdots, A_m$ *where* $p = m + 1$, *the vector space generated by* $Z_1, Z_2, \cdots, Z_p$ *is a commutative ideal, the vector space generated by* $A_1, \cdots A_m$ *is commutative and the matrix representation of* $\text{ad}\left(\sum_{k=1}^m t_k A_k\right)$ *restricted to* $\mathfrak{p}$ *is given by*

$$A(t) = \left[\text{ad}\sum_{k=1}^m t_k A_k\right]\Big|\mathfrak{p} = m!\begin{bmatrix} 0 & t_1 & t_2 & \cdots & t_{m-1} & t_m \\ & 0 & t_1 & t_2 & \ddots & t_{m-1} \\ & & 0 & t_1 & \ddots & \vdots \\ & & & 0 & \ddots & t_2 \\ & & & & \ddots & t_1 \\ & & & & & 0 \end{bmatrix}.$$

Then $L^2(N)$ *admits a band-limited sampling subspace with respect to the discrete uniform subgroup* $\exp\left(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p\right)\exp\left(\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_m\right)$.

The work is organized as follows. In Section 2, we present general well-known results of harmonic analysis on nilpotent Lie groups. Section 3 contains intermediate results leading to the proofs of Theorem 4, Corollary 5 and Example 6 which are given in Section 4. Finally, in Section 5 we provide a method for constructing other sampling sets from Γ by using automorphisms of the Lie group N.

2. Harmonic Analysis on Nilpotent Lie Groups

2.1. **Parametrization of Coadjoint Orbits.** Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra of dimension n. We say that $\mathfrak{n}$ has a rational structure [3] if there is a real basis $\{Z_1, \cdots, Z_n\}$ for the Lie algebra $\mathfrak{n}$ having rational structure constants and the rational span of the basis $\{Z_1, \cdots, Z_n\}$ denoted by $\mathfrak{n}_{\mathbb{Q}}$ provides a rational structure

such that $\mathfrak{n}$ is isomorphic to the vector space $\mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$. Let $\mathfrak{B} = \{Z_1, \cdots, Z_n\}$ be a basis for the Lie algebra $\mathfrak{n}$ such that for any $Z_i, Z_j \in \mathfrak{B}$, we have:

$$\left[Z_i, Z_j\right] = \sum_{k=1}^{n} c_{ijk} Z_k$$

and $c_{ijk} \in \mathbb{Q}$. We say that $\mathfrak{B}$ is a **strong Malcev basis** (see Page 10, [3]) if and only if for each $1 \leq j \leq n$ the real span of $\left\{Z_1, Z_2, \cdots, Z_j\right\}$ is an ideal of $\mathfrak{n}$. Now, let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ having a rational structure. The following result is taken from Corollary 5.1.10, [3]. Let $\{Z_1, \cdots, Z_n\}$ be a strong Malcev basis for the Lie algebra $\mathfrak{n}$. There exists a suitable integer q such that $\Gamma_q = \exp\left(q\mathbb{Z}Z_1\right) \cdots \exp\left(q\mathbb{Z}Z_n\right)$ is a discrete uniform subgroup of N (there is a compact set $K \subset G$ such that $\Gamma K = N$). Setting $X_k = qZ_k$ for $1 \leq k \leq n$, from now on, we fix $\{X_1, \cdots, X_n\}$ as a strong Malcev basis for the Lie algebra $\mathfrak{n}$ such that

$$\Gamma = \exp\left(\mathbb{Z}X_1\right) \cdots \exp\left(\mathbb{Z}X_n\right)$$

is a discrete uniform subgroup of N.

We shall next discuss the Plancherel theory for N. This theory is well exposed in [3] for nilpotent Lie groups. Let $\mathfrak{s}$ be a subset of $\mathfrak{n} = \log(N)$. For each linear functional $\lambda \in \mathfrak{n}^*$, we define the corresponding set

$$\mathfrak{s}\left(\lambda\right) = \{Z \in \mathfrak{n} : \lambda\left(\left[Z, X\right]\right) = 0 \text{ for every } X \in \mathfrak{s}\}.$$

Next, we consider a fixed strong Malcev basis $\mathcal{B}' = \{X_1, \cdots, X_n\}$ and we construct a sequence of ideals $\mathfrak{n}_1 \subseteq \mathfrak{n}_2 \subseteq \cdots \subseteq \mathfrak{n}_{n-1} \subseteq \mathfrak{n}$ where each ideal $\mathfrak{n}_k$ is spanned by $\{X_1, \cdots, X_k\}$. It is easy to see that the differential of the coadjoint action on λ at the identity is given by the matrix

$$\left[\langle \lambda, \left[X_j, X_k\right] \rangle\right]_{1 \leq j,k \leq n} = \left[\lambda\left(\left[X_j, X_k\right]\right)\right]_{1 \leq j,k \leq n}.$$

Defining the skew-symmetric matrix-valued function

(7)
$$\lambda \mapsto \mathbf{M}\left(\lambda\right) = \begin{bmatrix} \lambda\left[X_1, X_1\right] & \cdots & \lambda\left[X_1, X_n\right] \\ \vdots & \ddots & \vdots \\ \lambda\left[X_n, X_1\right] & \cdots & \lambda\left[X_n, X_n\right] \end{bmatrix}$$

on $\mathfrak{n}^*$, it is worth noting that $\mathfrak{n}\left(\lambda\right)$ is equal to the null-space of $\mathbf{M}\left(\lambda\right)$, if $\mathbf{M}\left(\lambda\right)$ is regarded as a linear operator acting on $\mathfrak{n}$ [16]. According to the orbit method [3], the unitary dual of N is in one-to-one correspondence with the set of coadjoint orbits in the dual of the Lie algebra. For each $\lambda \in \mathfrak{n}^*$ we define

(8)
$$\mathbf{e}\left(\lambda\right) = \{1 \leq k \leq n : \mathfrak{n}_k \not\subseteq \mathfrak{n}_{k-1} + \mathfrak{n}\left(\lambda\right)\}.$$

The set $\mathbf{e}\left(\lambda\right)$ collects all basis elements $\{X_i : i \in \mathbf{e}\left(\lambda\right)\} \subset \{X_1, X_2, \cdots, X_{n-1}, X_n\}$ such that if the elements are ordered such that $\mathbf{e}\left(\lambda\right) = \{\mathbf{e}_1\left(\lambda\right) < \cdots < \mathbf{e}_{2m}\left(\lambda\right)\}$ then the dimension of the manifold $\exp\left(\mathbb{R}X_{\mathbf{e}_1(\lambda)}\right) \cdots \exp\left(\mathbb{R}X_{\mathbf{e}_{2m}(\lambda)}\right) \cdot \lambda$ is equal to the dimension of the N-orbit of λ. Each element of the set $\mathbf{e}\left(\lambda\right)$ is called a **jump index** and clearly the cardinality of the set of jump indices $\mathbf{e}\left(\lambda\right)$ must be equal to the dimension of the coadjoint orbit of λ.

For each subset $\mathbf{e}^\circ \subseteq \{1, 2, \cdots, n\}$, the set

$$\Omega_{\mathbf{e}^\circ} = \{\lambda \in \mathfrak{n}^* : \mathbf{e}(\lambda) = \mathbf{e}^\circ\}$$

is algebraic and N-invariant [5]. Moreover, there exists a set of jump indices $\mathbf{e}$ such that $\Omega_{\mathbf{e}} = \Omega$ is a Zariski open set in $\mathfrak{n}^*$ which is invariant under the action of N (Theorem 3.1.6, [3].)

Put $\Omega = \Omega_{\mathbf{e}}$. We recall that a polarization subalgebra subordinated to the linear functional λ is a maximal subalgebra $\mathfrak{p}(\lambda)$ of $\mathfrak{n}^*$ satisfying the condition $[\mathfrak{p}(\lambda), \mathfrak{p}(\lambda)] \subseteq \ker \lambda$. Notice that if $\mathfrak{p}(\lambda)$ is a polarization subalgebra associated with the linear functional λ then $\chi(\exp X) = e^{2\pi i \lambda(X)}$ defines a character on $\exp(\mathfrak{p}(\lambda))$. It is also well-known that $\dim(\mathfrak{n}(\lambda)) = n - 2m$ and $\dim(\mathfrak{n}/\mathfrak{p}(\lambda)) = m$, and $\mathfrak{p}(\lambda) = \sum_{k=1}^{n} \mathfrak{n}_k (\lambda | \mathfrak{n}_k)$ (see Page 30, [3] and [16].)

Proposition 7. *If $\mathfrak{p}$ is a constant polarization for $\mathfrak{n}$ then it must be commutative.*

Proof. Let Ω be a Zariski open and N-invariant subset of $\mathfrak{n}^*$ such that $\mathfrak{p}$ is an ideal subordinated to every linear functional $\lambda \in \Omega$. First, observe that $\Omega \cap [\mathfrak{p}, \mathfrak{p}]^*$ is open in $[\mathfrak{p}, \mathfrak{p}]^*$. Next, for arbitrary $\ell \in \Omega \cap [\mathfrak{p}, \mathfrak{p}]^*$, by assumption $[\mathfrak{p}, \mathfrak{p}]$ is contained in the kernel of ℓ. Thus, $[\mathfrak{p}, \mathfrak{p}]$ must be a trivial vector space and it follows that $\mathfrak{p}$ is commutative. $\square$

The following result is established in Theorem 3.1.9, [3]

Proposition 8. *A cross-section for the coadjoint orbits in Ω is*

(9) $$\Sigma = \{\lambda \in \Omega : \lambda(Z_k) = 0 \text{ for all } k \in \mathbf{e} \}$$

2.2. **Unitary Dual and Plancherel Theory.** The setting in which we are studying sampling spaces requires the following ingredients:

(1) An explicit description of the irreducible representations occurring in the decomposition of the left regular representation of N.
(2) The Plancherel measure, and a formula for the Fourier (Plancherel) transform.
(3) A description of left-invariant multiplicity-free spaces.

2.2.1. *A Realization of the Irreducible Representations of N.* The following discussion is mainly taken from Chapter 6, [7]. Let G be a locally compact group, and let K be a closed subgroup of G. Let us define $q : G \to G/K$ to be the canonical quotient map and let φ be a unitary representation of the group K acting in some Hilbert space which we call $\mathbf{H}$. Next, let $\mathbf{K}_1$ be the set of continuous $\mathbf{H}$-valued functions f defined over G satisfying the following properties:

- The image of the support of f under the quotient map q is compact.
- $f(gk) = \left[\varphi(k)^{-1} f\right](g)$ for $g \in G$ and $k \in K$.

Clearly, G acts on the set $\mathbf{K}_1$ by left translation. Now, to simplify the presentation, let us suppose that G/K admits a G-invariant measure (this assumption is not always true.) However, since we are mainly dealing with unimodular groups, the assumption holds. First, we endow $\mathbf{K}_1$ with the following inner product: $\langle f, f' \rangle = \int_{G/K} \langle f(g), f'(g) \rangle_{\mathbf{H}} d(gK)$ for $f, f' \in \mathbf{K}_1$. Second, let $\mathbf{K}$ be the Hilbert completion of the space $\mathbf{K}_1$ with respect to this inner product. The translation operators extend to

unitary operators on $\mathbf{K}$ inducing the unitary representation $\mathrm{Ind}_K^G(\varphi)$ which acts on $\mathbf{K}$ as follows:

$$\left[\mathrm{Ind}_K^G(\varphi)(x)f\right](g) = f\left(x^{-1}g\right) \text{ for } f \in \mathbf{K}.$$

We notice that if φ is a character, then the Hilbert space $\mathbf{K}$ can be naturally identified with $L^2(G/K)$. The reader who is not familiar with these notions is invited to refer to Chapter 6 of the book of Folland [7] for a thorough presentation.

For each linear functional in the set Σ (see (9)), there is a corresponding unitary irreducible representation of N which is realized as acting in $L^2(\mathbb{R}^m)$ as follows. Define a character χ_λ on the normal subgroup $\exp(\mathfrak{p}(\lambda))$ such that $\chi_\lambda(\exp X) = e^{2\pi i \lambda(X)}$ for $X \in \mathfrak{p}(\lambda)$. In order to realize an irreducible representation corresponding to the linear functional λ, induce the character χ_λ as follows:

(10) $$\sigma_\lambda = \mathrm{Ind}_{P_\lambda}^N(\chi_\lambda), \text{ where } P_\lambda = \exp(\mathfrak{p}(\lambda)).$$

The induced representation σ_λ acts by left translations on the Hilbert space

(11) $$\mathbf{H}_\lambda = \left\{ f : N \longrightarrow \mathbb{C} : f(xy) = \chi_\lambda(y)^{-1} f(x) \text{ for } y \in P_\lambda \right.$$
$$\left. \text{and } \int_{N/P_\lambda} |f(x)|^2 \, d(xP_\lambda) < \infty \right\},$$

which is endowed with the following inner product:

$$\left\langle f, f' \right\rangle = \int_{N/P_\lambda} f(n)\overline{f'(n)} d(nP_\lambda).$$

Picking a cross-section in N for N/P_λ, since χ_λ is a character there is an obvious identification between $\mathbf{H}_\lambda$ and the Hilbert space $L^2(N/P_\lambda) = L^2(\mathbb{R}^m)$.

2.2.2. *The Plancherel Measure and the Plancherel Transform.* For a linear functional $\lambda \in \Omega$, put $\mathbf{e} = \{\mathbf{e}_1 < \mathbf{e}_2 < \cdots < \mathbf{e}_{2m}\}$ and define

(12) $$B(\lambda) = \left[\lambda\left[X_{\mathbf{e}_i}, X_{\mathbf{e}_j}\right]\right]_{1 \le i,j \le 2m}.$$

Then $B(\lambda)$ is a skew-symmetric invertible matrix of rank $2m$. Let $d\lambda$ be the Lebesgue measure on Σ which is parametrized by a Zariski subset of $\mathbb{R}^{n-2m}$. Put

$$d\mu(\lambda) = |\det B(\lambda)|^{1/2} \, d\lambda.$$

It is proved in Section 4.3, [3] that up to multiplication by a constant, the measure $d\mu(\lambda)$ is the Plancherel measure for N. The group Fourier transform $\mathcal{F}$ is an operator-valued bounded operator which is weakly defined on $L^2(N) \cap L^1(N)$ as follows:

(13) $$\sigma_\lambda(f) = \mathcal{F}(f)(\lambda) = \int_\Sigma f(n)\sigma_\lambda\left(n^{-1}\right) dn \text{ where } f \in L^2(N) \cap L^1(N).$$

Moreover, given $\mathbf{u}, \mathbf{v} \in L^2(\mathbb{R}^m)$, we have

$$\left\langle \sigma_\lambda(f)\mathbf{u}, \mathbf{v} \right\rangle = \int_\Sigma f(n)\left\langle \sigma_\lambda\left(n^{-1}\right)\mathbf{u}, \mathbf{v} \right\rangle dn.$$

Next, the Plancherel transform is a unitary operator

$$\mathcal{P} : L^2(N) \longrightarrow L^2\left(\Sigma, L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m), d\mu(\lambda)\right)$$

which is obtained by extending the Fourier transform to $L^2(N)$. This extension induces the equality

$$\|f\|_{L^2(N)}^2 = \int_\Sigma \left\|\widehat{f}(\sigma_\lambda)\right\|_{\mathcal{HS}}^2 d\mu(\lambda)$$

where $\widehat{f}(\sigma_\lambda) = \mathcal{P}f(\lambda)$. Let L be the left regular representation of the nilpotent group N. It is easy to check that for almost every $\lambda \in \Sigma$ (with respect to the Plancherel measure)

$$\left(\mathcal{P}L(n)\mathcal{P}^{-1}A\right)(\sigma_\lambda) = \sigma_\lambda(n) \circ A(\sigma_\lambda).$$

In other words, the Plancherel transform intertwines the regular representation with a direct integral of irreducible representations of N. The irreducible representations occurring in the decomposition are parametrized up to a null set by the manifold Σ and each irreducible representation occurs with infinite multiplicities in the decomposition.

2.3. Bandlimited Multiplicity-Free Spaces.

Given any measurable set $\mathbf{A} \subseteq \Sigma$, it is easily checked that the Hilbert space

$$\mathcal{P}^{-1}\left(L^2\left(\mathbf{A}, L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m)\right), d\mu(\lambda)\right)$$

is a left-invariant subspace of $L^2(N)$. Let us suppose that $\mathbf{A}$ is a **bounded** subset of Σ of positive Plancherel measure. Letting $|\mathbf{A}|$ be the Lebesgue measure of the set $\mathbf{A}$

$$(14) \qquad \mu(\mathbf{A}) = \int_{\mathbf{A}} |\det B(\lambda)|^{1/2} d\lambda \leq |\mathbf{A}| \ \sup\left\{|\det B(\lambda)|^{1/2} : \lambda \in \mathbf{A}\right\}.$$

Next, since $\lambda \mapsto |\det B(\lambda)|^{\frac{1}{2}}$ is a continuous function then $\mu(\mathbf{A})$ is finite. Fix a measurable field $(\mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$ of unit vectors in $L^2(\mathbb{R}^m)$. Put

$$(15) \qquad \mathbf{H_A} = \left\{ f \in L^2(N) : \mathcal{P}f(\lambda) = \begin{cases} \mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda) \text{ if } \lambda \in \mathbf{A} \\ 0 \otimes 0 \text{ if } \lambda \notin \mathbf{A} \end{cases} \text{ and} \\ (\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}} \text{ is a measurable} \\ \text{field of rank-one operators} \right\}.$$

Then $\mathbf{H_A}$ is a left-invariant, band-limited and multiplicity-free subspace of $L^2(N)$. Let $h \in \mathbf{H_A}$ such that the Plancherel transform of h is a measurable field of rank-one operators. More precisely, let us assume that

$$(16) \qquad \mathcal{P}h(\sigma_\lambda) = \widehat{h}(\sigma_\lambda) = \begin{cases} \mathbf{u}(\lambda) \otimes \mathbf{u}(\lambda) \text{ if } \lambda \in \mathbf{A} \\ 0 \otimes 0 \text{ if } \lambda \notin \mathbf{A} \end{cases}$$

and

$$\left[V_h^L(f)\right](\exp(X)) = \langle f, L(\exp(X))h\rangle = f * h^*(\exp(X))$$

where $h^*(x) = \overline{h\left(x^{-1}\right)}$ and $f * g(n) = \int_N f(m) g\left(m^{-1}n\right) dm$.

Proposition 9. *If $\mathbf{A}$ is a bounded subset of Σ of positive Plancherel measure and if h is as given in (16) then h is an admissible vector for the representation $(L, \mathbf{H_A})$.*

Proof. To check that h is well-defined as an element of the Hilbert space $\mathbf{H_A}$, it is enough to verify that

$$\|h\|_{L^2(N)}^2 = \int_{\mathbf{A}} \|\mathbf{u}(\lambda) \otimes \mathbf{u}(\lambda)\|_{\mathcal{HS}}^2 \, \mathrm{d}\mu(\lambda) = \mu(\mathbf{A})$$

is finite. Next, for any vector $f \in \mathbf{H_A}$, the square of the norm of the image of f under the map V_h^L is computed as follows:

$$\left\|V_h^L(f)\right\|_{L^2(N)}^2 = \int_{\mathbf{A}} \left\|\widehat{f}(\sigma_\lambda)(\mathbf{u}(\lambda) \otimes \mathbf{u}(\lambda))\right\|_{\mathcal{HS}}^2 \mathrm{d}\mu(\lambda)$$
$$= \int_{\mathbf{A}} \left\langle \widehat{f}(\sigma_\lambda)\mathbf{u}(\lambda), \widehat{f}(\sigma_\lambda)\mathbf{u}(\lambda)\right\rangle_{L^2(\mathbb{R}^m)} \mathrm{d}\mu(\lambda).$$

Letting $\widehat{f}(\sigma_\lambda) = \mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda)$ where $\mathbf{v}(\lambda)$ is in $L^2(\mathbb{R}^m)$, it follows that

$$\left\|V_h^L(f)\right\|_{L^2(N)}^2 = \int_{\mathbf{A}} \langle \mathbf{v}(\lambda), \mathbf{v}(\lambda)\rangle_{L^2(\mathbb{R}^m)} \, \mathrm{d}\mu(\lambda) = \|f\|_{L^2(N)}^2.$$

In other words, the map V_h^L defines an isometry from $\mathbf{H_A}$ into $L^2(N)$ and the representation $(L, \mathbf{H_A})$ which is a subrepresentation of the left regular representation of N is admissible. Thus, the vector h is an admissible vector. $\square$

It is also worth noting that h is convolution idempotent in the sense that $h = h * h^* = h^*$. Next, $V_h^L(\mathbf{H_A})$ is a left-invariant vector subspace of $L^2(N)$ consisting of continuous functions. Moreover, the projection onto the Hilbert space $V_h^L(\mathbf{H_A})$ is given by right convolution in the sense that $V_h^L(\mathbf{H_A}) = L^2(N) * h$.

In order to simplify our presentation, we shall naturally identify the Hilbert space $\mathbf{H_A}$ with $L^2(\mathbf{A} \times \mathbb{R}^m, \mathrm{d}\mu(\lambda)\,\mathrm{dt})$. This identification is given by the map

$$(\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}} \mapsto [\mathbf{v}(\lambda)](t) := \mathbf{v}(\lambda, t)$$

for any measurable field of rank-one operators $(\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$.

Lemma 10. *Let π be a unitary representation of a group N acting in a Hilbert space $\mathbf{H}_\pi$. Assume that π is admissible, and let h be an admissible vector for π. Furthermore, suppose that $\pi(\Gamma)h$ is a tight frame with frame bound C_h. Then the vector space $V_h(\mathbf{H}_\pi)$ is a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions and $V_h(\mathbf{H}_\pi)$ is a sampling space with sinc-type function $\frac{1}{C_h}V_h(h)$.*

Lemma 10 is proved in Proposition 2.54, [9]. This result establishes a connection between admissibility and sampling theories. This connection will play a central role in the proof of our main results. The following result is a slight extension of Proposition 2.61 [9], the proof given here is essentially inspired by the one given in the Monograph [9].

Lemma 11. *Let Γ be a discrete subgroup of N with positive co-volume Let π be a unitary representation of N acting in a Hilbert space $\mathbf{H}_\pi$. If the restriction of π to the discrete subgroup Γ is unitarily equivalent to a subrepresentation of the left regular representation of Γ then there exists a subspace of $L^2(N)$ which is a sampling space with respect to Γ. Moreover, if π is equivalent to the left regular representation of Γ*

then there exists a subspace of $L^2(N)$ *which is a sampling space with the interpolation property with respect to* Γ.

Proof. Let $T : \mathbf{H}_\pi \to \mathbf{H} \subset l^2(\Gamma)$ be a unitary map which is intertwining the restricted representation of π to Γ with a representation which is a subrepresentation of the left regular representation of the lattice Γ. Since Γ is a discrete group, the left regular representation of Γ is admissible. To see this, let κ be the sequence which is equal to one at the identity of Γ and zero everywhere else. By shifting the sequence κ by elements in Γ, we generate an orthonormal basis for the Hilbert space $l^2(\Gamma)$. Now, let $P : l^2(\Gamma) \to \mathbf{H}$ be an orthogonal projection. Next, the vector $\eta = T^{-1}(P(\kappa))$ is an admissible vector for $\pi|_\Gamma$ as well. We recall that $V_\eta^\pi(f) = \langle f, \pi(\cdot)\eta \rangle$. Let $N = A\Gamma$ where A is a set of finite measure with respect to the Haar measure of N. Without loss of generality, let us assume that a Haar measure for N is fixed so that $|A| = 1$. Then

$$\left\| V_\eta^\pi(f) \right\|_{L^2(N)}^2 = \int_N |\langle f, \pi(x)\eta \rangle|^2 \, dx$$

$$= \int_A \sum_{\gamma \in \Gamma} |\langle f, \pi(x\gamma)\eta \rangle|^2 \, dx$$

$$= \int_A \sum_{\gamma \in \Gamma} \left| \left\langle \pi\left(x^{-1}\right) f, \pi(\gamma)\eta \right\rangle \right|^2 \, dx.$$

Next, since $\pi(\Gamma)\eta$ is a Parseval frame in $\mathbf{H}_\pi$,

$$\sum_{\gamma \in \Gamma} \left| \left\langle \pi\left(x^{-1}\right) f, \pi(\gamma)\eta \right\rangle \right|^2 = \left\| \pi\left(x^{-1}\right) f \right\|_{\mathbf{H}_\pi}^2$$

and it follows that

$$\left\| V_\eta^\pi(f) \right\|_{L^2(N)}^2 = \int_A \left\| \pi\left(x^{-1}\right) f \right\|_{\mathbf{H}_\pi}^2 \, dx = \|f\|_{\mathbf{H}_\pi}^2 |A| = \|f\|_{\mathbf{H}_\pi}^2.$$

Thus η is a continuous wavelet for the representation π, $\pi(\Gamma)(\eta)$ is a Parseval frame, and the Hilbert space $V_\eta^\pi(\mathbf{H}_\pi)$ is a sampling space of $L^2(N)$ with respect to the lattice Γ. Now, for the second part, if we assume that π is equivalent to the left regular representation, then the operator P described above is just the identity map. Next, $\pi(\Gamma)\eta = \pi(\Gamma)\left(T^{-1}(\kappa)\right)$ is an orthonormal basis of $\mathbf{H}_\pi$ and $V_\eta(\eta)$ is a sinc-type function. It follows from Theorem 2.56 [9] that $V_\eta(\mathbf{H}_\pi)$ is a sampling space of $L^2(N)$ which has the interpolation property with respect to the lattice Γ. $\qquad \square$

Remark 12. *Let $\mathbf{H_A}$ be the Hilbert space of band-limited functions as described in (15). We recall that $\Gamma = \exp(\mathbb{Z}X_1) \cdots \exp(\mathbb{Z}X_n)$ is a discrete uniform subgroup of N. Since $\mathbf{H_A}$ is left-invariant, the regular representation of N admits a subrepresentation obtained by restricting the action of the left regular representation to the Hilbert space $\mathbf{H_A}$. Let us denote such a representation by $L_{\mathbf{H_A}}$. Furthermore, let $L_{\mathbf{H_A}, \Gamma}$ be the restriction of $L_{\mathbf{H_A}}$ to Γ. If the representation $L_{\mathbf{H_A}, \Gamma}$ is unitarily equivalent to a subrepresentation of the left regular representation of the discrete group Γ then according to arguments used in the proof of Lemma 11, there exists a vector η such that $V_\eta^L(\mathbf{H_A})$ is a sampling space of $L^2(N)$ with respect to the discrete uniform group Γ. In the present*

work, we are aiming to find conditions on the spectral set **A** *which guarantees that* $L_{\mathbf{H_A},\Gamma}$ *is unitarily equivalent to a subrepresentation of the left regular representation of the discrete group* Γ. *We shall also prove that under the assumptions given in Condition 3, it is possible to find* **A** *such that* $V_\eta^L(\mathbf{H_A})$ *is a sampling space of* $L^2(N)$ *with respect to the discrete uniform group* Γ.

3. Intermediate Results

Let us now fix assumptions and specialize the theory of harmonic analysis of nilpotent Lie groups to the class of groups being considered here.

Let N be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$ with rational structure constants such that $N = PM = \exp(\mathfrak{p})\exp(\mathfrak{m})$ where $\mathfrak{p}$ and $\mathfrak{m}$ are commutative Lie algebras, and $\mathfrak{p}$ is an ideal of $\mathfrak{n}$. We fix a strong Malcev basis

(17) $$\left\{ Z_1, \cdots, Z_p, A_1, \cdots, A_m \right\}$$

for $\mathfrak{n}$ such that $\left\{ Z_1, \cdots, Z_p \right\}$ is a basis for $\mathfrak{p}$ and $\{A_1, \cdots, A_m\}$ is a basis for $\mathfrak{m}$. Therefore, N is isomorphic to the semi-direct product group $P \rtimes M$ endowed with the multiplication law

$$\left(\exp Z, \exp A\right)\left(\exp Z', \exp A'\right) = \left(\exp\left(Z + e^{\mathrm{adA}}Z'\right), \exp\left(A + A'\right)\right).$$

Moreover it is assumed that

$$\Gamma = \exp\left(\sum_{k=1}^{p} \mathbb{Z}Z_k\right)\exp\left(\sum_{k=1}^{m} \mathbb{Z}A_k\right)$$

is a discrete uniform subgroup of N. Indeed, in order to ensure that Γ is a discrete uniform group, it is enough to pick $\{A_1, \cdots, A_m\}$ such that the matrix representation of $e^{\mathrm{adA_k}}|\mathfrak{p}$ with respect to the basis $\left\{ Z_1, \cdots, Z_p \right\}$ has entries in $\mathbb{Z}$.

If $\mathbf{M}(\lambda)$ is the skew-symmetric matrix described in (7) then

$$\mathbf{M}(\lambda) = \begin{bmatrix} 0 & \cdots & 0 & \lambda[Z_1, A_1] & \cdots & \lambda[Z_1, A_m] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda[Z_p, A_1] & \cdots & \lambda[Z_p, A_m] \\ \lambda[A_1, Z_1] & \cdots & \lambda[A_1, Z_p] & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda[A_m, Z_1] & \cdots & \lambda[A_m, Z_p] & 0 & \cdots & 0 \end{bmatrix}.$$

Regarding $\mathbf{M}(\lambda)$ as the matrix representation of a linear operator acting on $\mathfrak{n}$, we recall that the null-space of $\mathbf{M}(\lambda)$ corresponds to $\mathfrak{n}(\lambda)$. Now, let $\mathbf{M_k}(\lambda)$ be the matrix

obtained by retaining the first k columns of $\mathbf{M}(\lambda)$ (see illustration below)

$$\mathbf{M}(\lambda) = \underbrace{\underbrace{\underbrace{\begin{bmatrix} 0 & \cdots & 0 & \lambda[Z_1, A_1] & \cdots & \lambda[Z_1, A_m] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda[Z_p, A_1] & \cdots & \lambda[Z_p, A_m] \\ \lambda[A_1, Z_1] & \cdots & \lambda[A_1, Z_p] & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda[A_m, Z_1] & \cdots & \lambda[A_m, Z_p] & 0 & \cdots & 0 \end{bmatrix}}_{\mathbf{M}_1(\lambda)}}_{\mathbf{M}_p(\lambda)}}_{\mathbf{M}_{p+1}(\lambda)}}.$$

$$\underbrace{\phantom{\mathbf{M}(\lambda)=\mathbf{M}_n(\lambda)}}_{\mathbf{M}(\lambda)=\mathbf{M}_n(\lambda)}$$

Although $\mathbf{M}_0(\lambda)$ is not defined, we shall need to assume that $\operatorname{rank}(\mathbf{M}_0(\lambda)) = 0$. Put

$$X_1 = Z_1, \cdots, X_p = Z_p, X_{p+1} = A_1, \cdots, X_n = A_m.$$

Lemma 13. *Given $\lambda \in \mathfrak{n}^*$, the following holds true.*

$$\{1 \le k \le n : \operatorname{rank}(\mathbf{M}_k(\lambda)) > \operatorname{rank}(\mathbf{M}_{k-1}(\lambda))\} = \{1 \le k \le n : \mathfrak{n}_k \not\subseteq \mathfrak{n}_{k-1} + \mathfrak{n}(\lambda)\}.$$

Proof. First, assume that the rank of $\mathbf{M}_i(\lambda)$ is greater than the rank of $\mathbf{M}_{i-1}(\lambda)$. Then it is clear that X_i cannot be in the null-space of the matrix $\mathbf{M}(\lambda)$. Thus, $\mathfrak{n}_i = \mathfrak{n}_{i-1} + \mathbb{R}X_i \not\subseteq \mathfrak{n}_{i-1} + \mathfrak{n}(\lambda)$. Next, if $\mathfrak{n}_i \not\subseteq \mathfrak{n}_{i-1} + \mathfrak{n}(\lambda)$ and $\mathfrak{n}_i = \mathfrak{n}_{i-1} + \mathbb{R}X_i$ since the basis element X_i cannot be in $\mathfrak{n}(\lambda)$. Thus the rank of $\mathbf{M}_i(\lambda)$ is greater than the rank of $\mathbf{M}_{i-1}(\lambda)$ and the stated result is established. $\square$

It is proved in Theorem 3.1.9, [3] that there exist a Zariski open subset Ω of $\mathfrak{n}^*$ and a fixed set $\mathbf{e} \subset \{1, 2, \cdots, n\}$ such that the map

$$\lambda \mapsto \{1 \le k \le n : \operatorname{rank}(\mathbf{M}_k(\lambda)) > \operatorname{rank}(\mathbf{M}_{k-1}(\lambda))\} = \mathbf{e}$$

is constant, Ω is invariant under the coadjoint action of N and

$$\Sigma = \{\lambda \in \Omega : \lambda(X_k) = 0 \text{ for all } k \in \mathbf{e}\}$$

is an algebraic set which is a cross-section for the coadjoint orbits of N in Ω, as well as a parameterizing set for the unitary dual of N.

Lemma 14. *If $\mathfrak{p}$ is a constant polarization for $\mathfrak{n}$ then the set*

$$\{p+1, p+2, \cdots, p+m = n\}$$

is contained in $\mathbf{e}$ and $\operatorname{card}(\mathbf{e}) = 2m$.

Proof. Let $\lambda \in \Sigma$. Let us suppose that there exists $k \in \{p+1, p+2, \cdots, p+m = n\}$ such that k is not an element of the set $\mathbf{e}(\lambda) = \mathbf{e}$. Without loss of generality, since the algebra generated by the A_j is commutative, we may assume that $X_k = A_1$. Indeed for any permutation $\sigma \in S_m$, $\{Z_1, \cdots, Z_p, A_{\sigma(1)}, \cdots A_{\sigma(m)}\}$ is a Malcev basis for $\mathfrak{n}$. Since

rank $(\mathbf{M}_k(\lambda)) = $ rank $(\mathbf{M}_{k-1}(\lambda))$ are since the matrices $M_{k-1}(\lambda)$, and $M_k(\lambda)$ are given as shown below

$$\mathbf{M}_k(\lambda) = \overbrace{\begin{bmatrix} 0 & \cdots & 0 & \lambda[Z_1, A_1] \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda[Z_p, A_1] \\ \lambda[A_1, Z_1] & \cdots & \lambda[A_1, Z_p] & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda[A_m, Z_1] & \cdots & \lambda[A_m, Z_p] & 0 \end{bmatrix}}^{=\mathbf{M}_{k-1}(\lambda)},$$

it is clear that for all $\lambda \in \Sigma$, the last column of the matrix $\mathbf{M}_k(\lambda)$ is equal to zero. It follows that $\mathfrak{p} + \mathbb{R}X_k$ is a commutative algebra and $[\mathfrak{p} + \mathbb{R}X_k, \mathfrak{p} + \mathbb{R}X_k]$ is contained in the kernel of the linear functional $\lambda \in \Sigma$; contradicting the fact that $\mathfrak{p}$ is a maximal algebra satisfying the condition that $[\mathfrak{p}, \mathfrak{p}] \subseteq \ker \lambda$. The second part of the lemma is true because $\mathbf{M}(\lambda)$ is a skew symmetric rank of positive rank. $\square$

Remark 15. *From now on, we shall assume that $\mathfrak{p}$ is a constant polarization ideal for $\mathfrak{n}$. Since $P = \exp\mathfrak{p}$ is normal in N, it is clear that the dual of the Lie algebra $\mathfrak{p}$ is invariant under the coadjoint action of the commutative group M.*

Remark 16. *Let $\beta : \Sigma \times \mathbb{R}^m \to \mathbb{R}^p$ be the mapping defined by*

$$\beta(\lambda, t_1, \cdots, t_m) = \exp(t_1 A_1 + \cdots + t_m A_m) \cdot \lambda|\mathfrak{p}^*$$

where $\cdot$ stands for the coadjoint action. In vector-form, $\beta(\lambda, t_1, \cdots, t_m)$ is easily computed as follows. Let $\mathfrak{P}(A(t))$ be the transpose of the matrix representation of $e^{-\mathrm{ad}(\sum_{k=1}^m t_k A_k)|\mathfrak{p}}$ with respect to the ordered basis $\{Z_k : 1 \le k \le p\}$. We write

$$\mathfrak{P}(A(t)) = \left[e^{-\mathrm{ad}(\sum_{k=1}^m t_k A_k)|\mathfrak{p}}\right]^{\mathrm{T}}$$

and

$$\beta(\lambda, t_1, \cdots, t_m) \equiv \mathfrak{P}(A(t)) \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix} \quad \textit{where } \lambda = \sum_{k=1}^p f_k Z_k^*$$

$\{Z_k^* : 1 \le k \le p\}$ *is a dual basis to $\{Z_k : 1 \le k \le p\}$. We shall generally make no distinction between linear functionals and their representations as either row or column vectors. Secondly for any linear functional $\lambda = \sum_{k=1}^p f_k Z_k^* \in \Sigma$ since $\mathfrak{P}(A(t))$ is a unipotent matrix, the components of $\beta(\lambda, t_1, \cdots, t_m)$ are polynomials in the variables f_k where $k \notin \mathbf{e}$ and $t_1, \cdots t_m$.*

Example 17. *Let us suppose that $\mathfrak{p}$ is spanned with Z_1, Z_2, Z_3 and $\mathfrak{m}$ is spanned by A_1, A_2 such that*

$$[A_1, Z_3] = Z_1, [A_2, Z_2] = Z_1, [A_2, Z_3] = Z_2.$$

Then

$$[\operatorname{ad}(t_1A_1 + t_2A_2)|_{\mathfrak{p}}] = \begin{bmatrix} 0 & t_2 & t_1 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \mathfrak{P}(A(t)) = \begin{bmatrix} 1 & 0 & 0 \\ -t_2 & 1 & 0 \\ \frac{1}{2}t_2^2 - t_1 & -t_2 & 1 \end{bmatrix}.$$

Lemma 18. *Let $\lambda \in \Sigma$. $\mathfrak{p}$ is a polarizing ideal subordinated to the linear functional λ if and only if for any given $X \in \mathfrak{n}$, $\mathfrak{p}$ is also a polarizing ideal subordinated to the linear functional $\exp(X) \cdot \lambda$.*

Proof. Appealing to Proposition 1.3.6 in [3], $\mathfrak{p}$ is a polarizing algebra subordinated to λ if and only if $e^{\operatorname{ad}X}\mathfrak{p} = \mathfrak{p}$ is a polarizing algebra subordinated to $\exp(X) \cdot \lambda$.   □

Lemma 19. *If for each $\lambda \in \Sigma$, $\mathfrak{p} = \mathbb{R}$-span $\{Z_1, \cdots, Z_p\}$ is a commutative polarizing ideal which is subordinated to the linear functional λ then β defines a diffeomorphism between $\Sigma \times \mathbb{R}^m$ and its range.*

Proof. In order to prove this result, it is enough to show that β is a bijective smooth map with constant full rank (see Theorem 6.5, [15]). In order to establish this fact, we will need to derive a precise formula for the coadjoint action. Let us define $\beta^{\mathbf{e}} : \Sigma \times \mathbb{R}^{2m}$ such that

$$\beta^{\mathbf{e}}(\lambda, t_{\mathbf{e}_1}, \cdots, t_{\mathbf{e}_{2m}}) = \exp(t_{\mathbf{e}_1}X_{\mathbf{e}_1}) \cdots \exp(t_{\mathbf{e}_{2m}}X_{\mathbf{e}_{2m}}) \cdot \lambda$$

and $\{\mathbf{e}_1 < \mathbf{e}_2 < \cdots < \mathbf{e}_{2m}\} = \mathbf{e}$. For a fixed linear functional λ in the cross-section Σ, the map

$$(t_{\mathbf{e}_1}, \cdots, t_{\mathbf{e}_{2m}}) \mapsto \exp(t_{\mathbf{e}_1}X_{\mathbf{e}_1}) \cdots \exp(t_{\mathbf{e}_{2m}}X_{\mathbf{e}_{2m}}) \cdot \lambda$$

defines a diffeomorphism between $\mathbb{R}^{2m}$ and the N-orbit of λ which is a closed submanifold of the dual of the Lie algebra $\mathfrak{n}$. Next, since all orbits in Ω are 2m-dimensional manifolds, the map $\beta^{\mathbf{e}}$ is a bijection with constant full-rank. Thus, $\beta^{\mathbf{e}}$ defines a diffeomorphism between $\Sigma \times \mathbb{R}^{2m}$ and Ω. Next, there exist indices $i_1, \cdots, i_m \leq p$ such that for $g = \exp(t_{\mathbf{e}_1}X_{\mathbf{e}_1}) \cdots \exp(t_{\mathbf{e}_{2m}}X_{\mathbf{e}_{2m}})$ we have

$$g \cdot \lambda = \exp\left(\sum_{k=1}^{m} t_{i_k}Z_{i_k}\right)\exp\left(\sum_{j=1}^{m} s_jA_j\right) \cdot \lambda.$$

In order to compute the coadjoint action of N on the linear functional λ, it is quite convenient to identify $\mathfrak{n}^*$ with $\mathfrak{p}^* \times \mathfrak{m}^*$ via the map

$$\iota : \mathfrak{n}^* = \mathfrak{p}^* + \mathfrak{m}^* \to \mathfrak{p}^* \times \mathfrak{m}^*$$

which is defined as follows:

$$\iota(f_1 + f_2) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ where } f_1 \in \mathfrak{p}^* \text{ and } f_2 \in \mathfrak{m}^*.$$

Thus, for any linear functional $\lambda \in \Sigma$,

$$\iota(\lambda) = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

for some $f \in \mathfrak{p}$. Put

$$A(s) = \sum_{j=1}^{m} \left(s_j A_j \right) \in \mathfrak{m}, \ Z(t) = \sum_{k=1}^{m} \left(t_{i_k} Z_{i_k} \right) \in \mathfrak{p}$$

where $s = (s_1, \cdots, s_m), t = \left(t_{i_1}, \cdots, t_{i_m} \right)$, and let $\left[e^{-\mathrm{adA}(s)|\mathfrak{p}} \right]$ be the matrix representation of the linear map $e^{-\mathrm{adA}(s)|\mathfrak{p}}$ which is obtained by exponentiating $-\mathrm{adA}(s)$ restricted to the vector space $\mathfrak{p}$. Clearly, with the fixed choice of the Malcev basis described in (17), it is easy to check that (see Remark 16)

$$(18) \qquad \iota \left(\exp \left(A(s) \right) \cdot \lambda \right) = \left[\begin{array}{c} \mathfrak{P}(A(s)) f \\ 0 \end{array} \right]$$

and

$$(19) \qquad \iota \left(\exp \left(Z(t) \right) \cdot \lambda \right) = \left[\begin{array}{c} f \\ \sigma(t, f) \end{array} \right]$$

where $\left(t_{i_1}, \cdots, t_{i_m} \right) \mapsto \sigma \left(t_{i_1}, \cdots, t_{i_m}, f \right)$ is an $m \times 1$ vector-valued function. Putting (18) and (19) together,

$$\exp \left(Z(t) \right) \exp \left(A(s) \right) \cdot \lambda = \left[\begin{array}{c} \mathfrak{P}(A(s)) f \\ \sigma(t, \mathfrak{P}(A(s)) f) \end{array} \right].$$

In order to compute the Jacobian of the map

$$(20) \qquad \left(f, s_1, \cdots, s_m, t_{i_1}, \cdots, t_{i_m} \right) \mapsto \left[\begin{array}{c} \mathfrak{P}(A(s)) f \\ \sigma(t, \mathfrak{P}(A(s)) f) \end{array} \right]$$

at the point (f, s, t), we set

$$\left[\begin{array}{c} \mathfrak{P}(A(s)) f \\ \sigma(t, \mathfrak{P}(A(s)) f) \end{array} \right] = \left[\begin{array}{c} \beta_1^{\mathbf{e}}(f, s) \\ \vdots \\ \beta_p^{\mathbf{e}}(f, s) \\ \beta_{p+1}^{\mathbf{e}}(f, s, t) \\ \vdots \\ \beta_n^{\mathbf{e}}(f, s, t) \end{array} \right]$$

where

$$\mathfrak{P}(A(s)) f = \left[\begin{array}{c} \beta_1^{\mathbf{e}}(f, s) \\ \vdots \\ \beta_p^{\mathbf{e}}(f, s) \end{array} \right] \text{ and } \sigma(t, \mathfrak{P}(A(s)) f) = \left[\begin{array}{c} \beta_{p+1}^{\mathbf{e}}(f, s, t) \\ \vdots \\ \beta_n^{\mathbf{e}}(f, s, t) \end{array} \right].$$

Now, let $\psi : \Sigma \times \mathbb{R}^{2m} \to \mathbb{R}^{\dim \Sigma} \times \mathbb{R}^{2m}$ such that

$$(\lambda, z) \mapsto \psi((\lambda, z)) = ((\ell_1, \cdots, \ell_{\dim \Sigma}), z) \text{ and } \ell = (\ell_1, \cdots, \ell_{\dim \Sigma}).$$

Then the pair $\left(\Sigma \times \mathbb{R}^{2m}, \psi \right)$ is a smooth chart around the point (λ, z). Computing the Jacobian of (20) in local coordinates, we obtain the following matrix

(21)
$$\left[\begin{array}{cc} A\left(\ell, s \right) & \\ C\left(\ell, s, t \right) & D\left(\ell, s, t \right) \end{array} \right]$$

where $(\ell, s) \mapsto A\left(\ell, s \right)$ is a matrix-valued function of order p given by

$$\left[\begin{array}{ccccccc} \dfrac{\partial \left(\beta_1^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial \ell_1} & \cdots & \dfrac{\partial \left(\beta_1^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial \ell_{\dim \Sigma}} & \dfrac{\partial \left(\beta_1^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial s_1} & \cdots & \dfrac{\partial \left(\beta_1^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial s_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \left(\beta_p^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial \ell_1} & \cdots & \dfrac{\partial \left(\beta_p^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial \ell_{\dim \Sigma}} & \dfrac{\partial \left(\beta_p^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial s_1} & \cdots & \dfrac{\partial \left(\beta_p^{\mathbf{e}} \psi^{-1}\left(\ell, s \right) \right)}{\partial s_m} \end{array} \right];$$

$(\ell, s, t) \mapsto C\left(\ell, s, t \right)$ is a $m \times p$ matrix-valued function which is equal to

$$\left[\begin{array}{ccccc} \dfrac{\partial \left(\beta_{p+1}^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial \ell_1} & \cdots & \dfrac{\partial \left(\beta_{p+1}^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial s_1} & \cdots & \dfrac{\partial \left(\beta_{p+1}^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial s_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \dfrac{\partial \left(\beta_n^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial \ell_1} & \cdots & \dfrac{\partial \left(\beta_n^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial s_1} & \cdots & \dfrac{\partial \left(\beta_n^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial s_m} \end{array} \right]$$

and finally

$$D\left(\ell, s, t \right) = \left[\begin{array}{ccc} \dfrac{\partial \left(\beta_{p+1}^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial t_1} & \cdots & \dfrac{\partial \left(\beta_{p+1}^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial t_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \left(\beta_n^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial t_1} & \cdots & \dfrac{\partial \left(\beta_n^{\mathbf{e}} \psi^{-1}\left(\ell, s, t \right) \right)}{\partial t_m} \end{array} \right]$$

is of order m. Now, since

$$\det \left[\begin{array}{cc} A\left(\ell, s \right) & \\ C\left(\ell, s, t \right) & D\left(\ell, s, t \right) \end{array} \right] = \det A\left(\ell, s \right) \times \det D\left(\ell, s, t \right) \neq 0$$

and because the Jacobian of β is the submatrix $A\left(\ell, s \right)$, it follows that its determinant does not vanish. Thus, the Jacobian of the map β has constant full rank as well. In order to establish that is a diffeomorphism, we appeal to the fact that

$$\iota \left(\beta \left(\lambda, s_1, \cdots, s_m \right) \right) = \mathfrak{P}\left(A(s) \right) f$$

and

$$\left(f, s_1, \cdots, s_m, t_{i_1}, \cdots, t_{i_m} \right) \mapsto \left[\begin{array}{c} \mathfrak{P}\left(A(s) \right) f \\ \sigma \left(t, \mathfrak{P}\left(A(s) \right) f \right) \end{array} \right]$$

is a bijection. Thus it is clear that β is smooth bijective with constant full-rank. Thus, it is a diffeomorphism. This completes the proof. $\square$

Remark 20. *For $\lambda \in \Sigma$ there exist real numbers f_k such that $\lambda = \sum_{k \notin \mathbf{e}} f_k Z_k^* = f_{k_1} Z_{k_1}^* + \cdots + f_{k_{n-2m}} Z_{k_{n-2m}}^*$. Thus, defining*

$$j : f_{k_1} Z_{k_1}^* + \cdots + f_{k_{n-2m}} Z_{k_{n-2m}}^* \mapsto \left(f_{k_1}, \cdots, f_{k_{n-2m}} \right),$$

it is clear that the unitary dual of the Lie group N is parametrized by $j(\Sigma)$ which is a Zariski open subset of $\mathbb{R}^{\dim \Sigma}$. Although this is an abuse of notation, we shall make no distinction between $j(\Sigma)$ and Σ.

An example is now in order.

Example 21. *Let $\mathfrak{n}$ be a nilpotent Lie algebra spanned by*

$$Z_1 = X_1, Z_2 = X_2, Z_4 = X_3, A_1 = X_4$$

with non-trivial Lie brackets

$$[X_4, X_2] = 2X_1 \ and \ [X_4, X_3] = 2X_2.$$

Letting

$$\lambda = \lambda_1 X_1^* + \lambda_2 X_2^* + \lambda_3 X_3^* + \alpha X_4^* \in \mathfrak{n}^*,$$

the skew symmetric matrix-valued function $\mathbf{M}$ is computed as follows:

$$\mathbf{M}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda(X_1) \\ 0 & 0 & 0 & -2\lambda(X_2) \\ 0 & 2\lambda(X_1) & 2\lambda(X_2) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_1 \\ 0 & 0 & 0 & -2\lambda_2 \\ 0 & 2\lambda_1 & 2\lambda_2 & 0 \end{bmatrix}.$$

We check that

$$\lambda \mapsto \mathbf{e}(\lambda) = \begin{cases} \{2,4\} & if \ \lambda(X_1) \neq 0 \\ \{3,4\} & if \ \lambda(X_1) = 0 \ and \ \lambda(X_2) \neq 0 \\ \{\} & if \ \lambda(X_1) = 0 \ and \ \lambda(X_2) = 0 \end{cases}.$$

Next,

$$\Omega = \Omega_{\{2,4\}} = \{\lambda = \lambda_1 X_1^* + \lambda_2 X_2^* + \lambda_3 X_3^* + \alpha X_4^* \in \mathfrak{n}^* : \lambda_1 \neq 0\}$$

is Zariski open, N-invariant and

$$\Sigma = \left\{ \lambda \in \Omega_{\{2,4\}} : \lambda(X_2) = \lambda(X_4) = 0 \right\} = (\mathbb{R} - \{0\}) \times \mathbb{R}$$

is a cross-section for the coadjoint orbits in $\Omega_{\{2,4\}}$. We note that for every linear functional $\lambda \in \Sigma$, the ideal

$$\mathfrak{p} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3$$

is a polarizing algebra subordinated to λ. Next,

$$\mathfrak{P}(A(t)) = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ \frac{1}{2}t^2 & -t & 1 \end{bmatrix}$$

and

$$\beta(\lambda, t) = \beta(\lambda_1, \lambda_3, t) = (\lambda_1) X_1^* - (2t\lambda_1) X_2^* + \left(2\lambda_1 t^2 + \lambda_3 \right) X_3^* = \left(\lambda_1, -2t\lambda_1, 2\lambda_1 t^2 + \lambda_3 \right)$$

defines a diffeomorphism between the sets $\Sigma \times \mathbb{R}$ and $\Omega_{\{2,4\}}$. Additionally, it is clear that $\beta(\Sigma \times \mathbb{R})$ is a Zariski open subset of $\mathbb{R}^3$ and

$$\beta(\Sigma \times \mathbb{R}) = (\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R}.$$

Remark 22. *For each $\lambda \in \Sigma$, the corresponding unitary irreducible representation σ_λ of N (see (10)) is obtained by inducing the character χ_λ of the normal subgroup P which is defined as follows:*

$$\chi_\lambda\left(\exp\left(t_1 Z_1 + \cdots + t_p Z_p\right)\right) = e^{2\pi i \langle \lambda, t_1 Z_1 + \cdots + t_p Z_p \rangle} = e^{2\pi i \lambda \left(t_1 Z_1 + \cdots + t_p Z_p\right)}.$$

Since $\exp(\mathbb{R}A_1 + \cdots + \mathbb{R}A_m)$ is a cross-section for N/P in N, we shall realize the unitary representation σ_λ as acting on the Hilbert space $L^2(N/P) = L^2(\mathbb{R}^m)$. Following the discussion in Subsection 2.2.1, it is easy to see that if $x = (x_1, \cdots, x_m)$,

$$A(a) = a_1 A_1 + \cdots + a_m A_m, \ \ Z(t) = t_1 Z_1 + \cdots + t_p Z_p$$

and $h \in L^2(\mathbb{R}^m)$, then for every linear functional $\lambda \in \Sigma$

$$(22) \quad [\sigma_\lambda(\exp(Z(t))\exp(A(a)))h](x) = e^{2\pi i \left\langle \lambda, e^{-\text{ad}A(x)} Z(t) \right\rangle} h(x_1 - a_1, \cdots, x_m - a_m).$$

We remark that although the Plancherel measure for an arbitrary nilpotent Lie group has already been computed in general form in the book of Corwin and Greenleaf [3], in order to prove the main results stated in the introduction, we will need to establish a connection between the Plancherel measure of N and the determinant of the Jacobian of the map β. To make this connection as transparent and as clear as possible, we shall need the following lemma.

Lemma 23. *Let $J_\beta(\lambda, s_1, \cdots, s_m)$ be the Jacobian of the smooth map β defined in (17). The Plancherel measure of N is up to multiplication by a constant equal to*

$$(23) \qquad\qquad d\mu(\lambda) = \left|\det J_\beta(\lambda, 0)\right| d\lambda$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{\dim \Sigma}$.

Proof. Since the set of smooth functions of compact support is dense in $L^2(N)$, it suffices to show that for any smooth function $\mathbf{F}$ of compact support on the group N,

$$\int_\Sigma \left\|\widehat{\mathbf{F}}(\sigma_\lambda)\right\|^2_{\mathcal{HS}(L^2(\mathbb{R}^m))} \left|\det J_\beta(\lambda, 0)\right| d\lambda = \|\mathbf{F}\|^2_{L^2(N)}.$$

In order to simplify our presentation, we shall identify the set $N = PM$ with $\mathbb{R}^p \times \mathbb{R}^m$ via the map

$$\exp\left(t_1 Z_1 + \cdots + t_p Z_p\right) \exp(a_1 A_1 + \cdots + a_m A_m) \mapsto \left(t_1, \cdots, t_p, a_1, \cdots, a_m\right) = (t, a).$$

For any smooth function $\mathbf{F}$ of compact support on the group N, the operator $\widehat{\mathbf{F}}(\sigma_\lambda)$ (see (13)) is defined on $L^2(\mathbb{R}^m)$ as follows. For $\phi \in L^2(\mathbb{R}^m)$ we have

$$(24) \qquad \left[\widehat{\mathbf{F}}(\sigma_\lambda)\phi\right](x) = \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \mathbf{F}(t,a)\left[(\sigma_\lambda(t,a))\phi\right](x)\,da\,dt$$

$$(25) \qquad = \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \mathbf{F}(t,a)\,e^{2\pi i\left\langle \lambda, e^{-adA(x)}Z(t)\right\rangle}\phi(x-a)\,da\,dt$$

$$(26) \qquad = \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \mathbf{F}(t,a)\,e^{2\pi i\langle \exp(A(x))\cdot\lambda, Z(t)\rangle}\phi(x-a)\,da\,dt.$$

Next, we recall that $\mathfrak{P}(A(x))\,f = \left[e^{-adA(x)|\mathfrak{p}}\right]^{\mathrm{T}} f$ and

$$\iota(\exp(A(x))\cdot\lambda) = \left[\begin{array}{c} \mathfrak{P}(A(x))\,f \\ 0 \end{array}\right] \text{ where } \iota(\lambda) = \left[\begin{array}{c} f \\ 0 \end{array}\right].$$

Next,

$$\left[\widehat{\mathbf{F}}(\sigma_\lambda)\phi\right](x) = \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \mathbf{F}(t,a)\,e^{2\pi i\langle \exp(A(x))\cdot\lambda, Z(t)\rangle}\phi(x-a)\,da\,dt$$

$$= \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \mathbf{F}(t,a)\,e^{2\pi i\langle \beta(\lambda,x), Z(t)\rangle}\phi(x-a)\,da\,dt$$

$$= \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \mathbf{F}(t,x-a)\,e^{2\pi i\langle \beta(\lambda,x), Z(t)\rangle}\phi(a)\,da\,dt$$

$$= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^m} \mathbf{F}(t,x-a)\,e^{2\pi i\langle \beta(\lambda,x), Z(t)\rangle}\,dt\right)\phi(a)\,da.$$

Thus, $\widehat{\mathbf{F}}(\sigma_\lambda)$ is an integral operator on $L^2(\mathbb{R}^m)$ with kernel $K_{\lambda,\mathbf{F}}$ given by

$$(27) \qquad K_{\lambda,\mathbf{F}}(x,a) = \int_{\mathbb{R}^p} \mathbf{F}(t,x-a)\,e^{2\pi i\langle \beta(\lambda,x), Z(t)\rangle}\,dt.$$

Now, let $\mathfrak{F}_1$ be the partial Euclidean Fourier transform in the direction of t. It is clear that

$$K_{\lambda,\mathbf{F}}(x,a) = [\mathfrak{F}_1\mathbf{F}](\beta(\lambda,x), x-a).$$

Additionally, the square of the Hilbert-Schmidt norm of the operator $\widehat{\mathbf{F}}(\sigma_\lambda)$ is given by

$$\left\|\widehat{\mathbf{F}}(\sigma_\lambda)\right\|^2_{\mathcal{HS}(L^2(\mathbb{R}^m))} = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left|K_{\lambda,\mathbf{F}}(x,a)\right|^2 dx\,da$$

$$= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |\mathfrak{F}_1\mathbf{F}(\beta(\lambda,x), x-a)|^2\,dx\,da$$

$$= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |\mathfrak{F}_1\mathbf{F}(\beta(\lambda,x), a)|^2\,dx\,da.$$

Observing that

$$\iota(\beta(\lambda, x+t)) = \mathfrak{P}(A(x))\,\mathfrak{P}(A(t))\,f,$$

the components of $\beta(\lambda, x+t)$ may be computed by multiplying a unipotent matrix by the matrix representation of $\beta(\lambda, t)$, the determinant of the Jacobian of the map β at

(λ, x) is then given by

$$\det J_\beta\left((\lambda, x + t)\right) = \det\left(\left[e^{(-\mathrm{adx}_1 A_1 - \cdots - \mathrm{adx}_m A_m)|_{\mathfrak{p}}}\right]\right)\det J_\beta\left(\lambda, t\right)$$
$$= 1 \times \det J_\beta\left(\lambda, t\right).$$

It follows that

$$\det J_\beta\left(\lambda, x\right) = \det J_\beta\left(\lambda, 0\right) = \mathbf{P}\left(\lambda\right)$$

where $\mathbf{P}\left(\lambda\right)$ is a polynomial in the coordinates of λ. Next,

$$\int_\Sigma \left\|\widehat{\mathbf{F}}\left(\sigma_\lambda\right)\right\|^2_{\mathcal{H}\mathrm{S}(\mathrm{L}^2(\mathbb{R}^m))}\left|\det J_\beta\left(\lambda, 0\right)\right| d\lambda$$
$$= \int_\Sigma \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\left|\left(\mathfrak{F}_1\mathbf{F}\right)\left(\beta\left(\lambda, x\right), a\right)\right|^2 da\ dx\ \left|\det J_\beta\left(\lambda, 0\right)\right|\right)d\lambda$$
$$= \int_\Sigma \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\left|\left(\mathfrak{F}_1\mathbf{F}\right)\left(\beta\left(\lambda, x\right), a\right)\right|^2 da\ dx\ \left|\det J_\beta\left(\lambda, 0\right)\right|\ d\lambda$$
$$= \int_{\Sigma \times \mathbb{R}^m}\int_{\mathbb{R}^m}\left|\left(\mathfrak{F}_1\mathbf{F}\right)\left(\beta\left(\lambda, x\right), a\right)\right|^2 da\ \left|\det J_\beta\left(\lambda, 0\right)\right|\ d\left(\lambda, x\right)$$
$$= \int_{\Sigma \times \mathbb{R}^m}\int_{\mathbb{R}^m}\left|\left(\mathfrak{F}_1\mathbf{F}\right)\left(\beta\left(\lambda, x\right), a\right)\right|^2 da\ d\left(\beta\left(\lambda, x\right)\right)$$
$$= \int_\Omega\int_{\mathbb{R}^m}\left|\left(\mathfrak{F}_1\mathbf{F}\right)\left(z, a\right)\right|^2 da\ dz.$$

Next, appealing to Plancherel's theorem

$$\int_\Sigma \left\|\widehat{\mathbf{F}}\left(\sigma_\lambda\right)\right\|^2_{\mathcal{H}\mathrm{S}(\mathrm{L}^2(\mathbb{R}^m))}\left|\det J_\beta\left(\lambda, 0\right)\right| d\lambda = \int_{\mathbb{R}^p}\int_{\mathbb{R}^m}\left|\mathbf{F}\left(z, a\right)\right|^2 dadz = \left\|\mathbf{F}\right\|^2_{\mathrm{L}^2(\mathrm{N})}.$$

$\square$

We shall now define a transform which plays an important role in proving our main results. Let $\mathbf{A}$ be a $d\mu$-measurable subset of $\mathbb{R}^{n-2m}$. Define (in a formal way) the map

$$J_{\mathbf{A}} : \mathrm{L}^2\left(\mathbf{A} \times \mathbb{R}^m, d\mu\left(\lambda\right)\right) \to \mathrm{l}^2\left(\Gamma\right)$$

such that for $l = (l_1, \cdots, l_m) \in \mathbb{Z}^m$,

$$Z\left(k\right) = \sum_{j=1}^p k_j Z_j \in \sum_{j=1}^p \mathbb{Z}Z_j \text{ and } A\left(l\right) = \sum_{j=1}^m l_j A_j \in \sum_{j=1}^m \mathbb{Z}A_j$$

we have

$$(28) \quad \left[J_{\mathbf{A}}F\right]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right) = \int_{\mathbf{A}}\int_{[0,1)^m} F\left(\lambda, t - l\right)e^{2\pi i\langle\beta(\lambda, t), Z(k)\rangle}dt\ d\mu\left(\lambda\right)$$

where $k \in \mathbb{Z}^m$.

Let $\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}$ be the Fourier transform defined on $\mathrm{L}^2\left(\mathbb{R}^p/\mathbb{Z}^p\right)$. Let 1_X denotes the indicator function for a given set X.

Proposition 24. *Assume that* $\mathbf{A}$ *is a* $d\mu$-*measurable bounded subset of* $\mathbb{R}^{n-2m}$. *If* $H \in \mathrm{L}^2\left(\mathbf{A} \times \mathbb{R}^m\right)$ *is a smooth function of compact support, then* $J_{\mathbf{A}}H \in \mathrm{l}^2\left(\Gamma\right)$.

Proof. First, observe that

$$[J_{\mathbf{A}}H]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right) = \int_{\mathbf{A}}\int_{[0,1)^m} H\left(\lambda, t-l\right)\ e^{2\pi i\langle\beta(\lambda,t),Z(k)\rangle}dt\ d\mu\left(\lambda\right)$$

and for a fixed $\sum_{j=1}^p k_jZ_j$, the sequence

$$([J_{\mathbf{A}}H]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right))_{l\in\mathbb{Z}^m}$$

has compact support. Making the change of variable $s = t - l$ we obtain that

$$[J_{\mathbf{A}}H]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right) = \int_{\mathbf{A}}\int_{[0,1)^m-l} H\left(\lambda, s\right)\ e^{2\pi i\langle\beta(\lambda,s+l),Z(k)\rangle}ds\ d\mu\left(\lambda\right).$$

Next, since

$$\beta\left(\lambda, s+l\right) = \exp\left(A\left(s\right)\right)\exp\left(A\left(l\right)\right)\cdot\lambda$$

and

$$d\mu\left(\lambda\right) = \left|\det J_{\beta(\lambda,0)}\right|d\lambda$$

it follows that

$$[J_{\mathbf{A}}H]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right) = \int_{\mathbf{A}\times([0,1)^m-l)} H\left(\lambda, s\right)\ e^{2\pi i\langle\beta(\lambda,s+l),Z(k)\rangle}\left|\det J_{\beta(\lambda,0)}\right|\ d\left(\lambda, s\right)$$

$$= \int_{\mathbf{A}\times([0,1)^m-l)} H\left(\lambda, s\right)\ e^{2\pi i\left\langle\beta(\lambda,s),e^{-adA(l)}Z(k)\right\rangle}\left|\det J_{\beta(\lambda,0)}\right|\ d\left(\lambda, s\right).$$

The last equality above is due to the equation

$$\left\langle\exp\left(\sum_{j=1}^m s_jA_j\right)\cdot\omega, Z\left(k\right)\right\rangle = \left\langle\omega, e^{-ad\left(\sum_{j=1}^m s_jA_j\right)}Z\left(k\right)\right\rangle \text{ for } \omega\in\Sigma.$$

Next, the change of variable $r = \beta\left(\lambda, s\right)$ yields

$$[J_{\mathbf{A}}H]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right) = \int_{\beta(\mathbf{A}\times([0,1)^m-l))}\left(H\circ\beta^{-1}\right)(r)\ e^{2\pi i\left\langle r,e^{-adA(l)}Z(k)\right\rangle}\ dr$$

where dr is the Lebesgue measure on $\mathbb{R}^p$. Next for each l, we write $\beta\left(\mathbf{A}\times\left([0,1)^m-l\right)\right)$ as a finite disjoint union of subsets of $\mathbb{R}^p$; each contained in a fundamental domain of $\mathbb{Z}^p$ as follows:

$$\beta\left(\mathbf{A}\times\left([0,1)^m-l\right)\right) = \overset{\cdot}{\underset{j\in J(H,l)}{\bigcup}}\left(K_{\mathbf{A},l}+j\right) \qquad \text{(where } J(H,l)\subset\mathbb{Z}^p\text{).}$$

Letting $1_{K_{\mathbf{A},l}+j}$ be the indicator function of the set $K_{\mathbf{A},l}+j$, we obtain

$$[J_{\mathbf{A}}H]\left(\exp\left(Z\left(k\right)\right)\exp\left(A\left(l\right)\right)\right) = \sum_{j\in J(H,l)}\int_{K_{\mathbf{A},l}+j}\left(H\circ\beta^{-1}\right)(r)\ e^{2\pi i\left\langle r,e^{-adA(l)}Z(k)\right\rangle}\ dr$$

$$= \sum_{j\in J(H,l)}\left[\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\left(\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}\right)\right]\left(e^{-adA(l)}Z\left(k\right)\right)$$

where for each $l\in\mathbb{Z}^m$

$$\sum_{j\in J(H,l)}\left[\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\left(\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}\right)\right]\left(e^{-adA(l)}Z\left(k\right)\right)$$

is a finite sum of Fourier transforms of smooth functions of compact support. In order to avoid cluster of notation, we set

$$(29) \qquad e^{-\mathrm{ad}A(l)}Z(k) = e^{-\mathrm{ad}A(l)}\left(\sum_{j=1}^{p} k_j Z_j\right) = a_l(k).$$

Finally,

$$\|[J_{\mathbf{A}}H]\|_{l^2(\Gamma)}^2 = \sum_{(k,l)\in\mathbb{Z}^p\times\mathbb{Z}^m} |J_{\mathbf{A}}H(\exp(Z(k))\exp(A(l)))|^2$$

$$= \sum_{k\in\mathbb{Z}^p}\sum_{l\in\mathbb{Z}^m} |J_{\mathbf{A}}H(\exp(Z(k))\exp(A(l)))|^2$$

$$= \sum_{k\in\mathbb{Z}^p}\sum_{l\in F}\left|\sum_{j\in J(H,l)} \left[\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\left(\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l+j}}\right)\right](a_l(k))\right|^2$$

where F is a finite subset of $\mathbb{Z}^m$ and

$$\|[J_{\mathbf{A}}H]\|_{l^2(\Gamma)}^2 = \sum_{l\in F}\sum_{k\in\mathbb{Z}^p}\left|\sum_{j\in J(H,l)} \left[\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\left(\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l+j}}\right)\right](a_l(k))\right|^2.$$

$$\leq \sum_{l\in F}\sum_{k\in\mathbb{Z}^p}\left(\sum_{j\in J(H,l)}\left|\left(\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\left(\overbrace{\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l+j}}}^{=\Theta_{H,l,j}}\right)\right)(a_l(k))\right|\right)^2.$$

Put

$$S_H(l) = \sum_{k\in\mathbb{Z}^p}\left(\sum_{j\in J(H,l)}\left|\left(\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\Theta_{H,l,j}\right)(a_l(k))\right|\right)^2.$$

Expanding the inner sum

$$\left(\sum_{j\in J(H,l)}\left|\left(\mathfrak{F}_{\mathbb{R}^p/\mathbb{Z}^p}\Theta_{H,l,j}\right)(a_l(k))\right|\right)^2$$

we obtain that

$$
\begin{aligned}
\mathrm{S}_{\mathrm{H}}\left(\mathrm{l}\right) = {}& \sum_{\mathrm{k}\in\mathbb{Z}^{\mathrm{p}}}\sum_{\mathrm{j}\in\mathrm{J}(\mathrm{H},\mathrm{l})}\left|\left(\mathfrak{F}_{\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}}}\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}\right)\left(\mathrm{a}_{\mathrm{l}}\left(\mathrm{k}\right)\right)\right|^{2} \\
& + 2\sum_{\mathrm{j}\neq\mathrm{j}'\text{and }\mathrm{j},\mathrm{j}'\in\mathrm{J}(\mathrm{H},\mathrm{l})}\sum_{\mathrm{k}\in\mathbb{Z}^{\mathrm{p}}}\left|\left(\mathfrak{F}_{\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}}}\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}\right)\left(\mathrm{a}_{\mathrm{l}}\left(\mathrm{k}\right)\right)\left(\mathfrak{F}_{\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}}}\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}'}\right)\left(\mathrm{a}_{\mathrm{l}}\left(\mathrm{k}\right)\right)\right| \\
= {}& \sum_{\mathrm{j}\in\mathrm{J}(\mathrm{H},\mathrm{l})}\left\|\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}\right\|^{2} \\
& + 2\sum_{\mathrm{j}\neq\mathrm{j}'\text{and }\mathrm{j},\mathrm{j}'\in\mathrm{J}(\mathrm{H},\mathrm{l})}\sum_{\mathrm{k}\in\mathbb{Z}^{\mathrm{p}}}\left|\left(\mathfrak{F}_{\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}}}\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}\right)\left(\mathrm{a}_{\mathrm{l}}\left(\mathrm{k}\right)\right)\times\left(\mathfrak{F}_{\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}}}\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}'}\right)\left(\mathrm{a}_{\mathrm{l}}\left(\mathrm{k}\right)\right)\right| \\
= {}& \sum_{\mathrm{j}\in\mathrm{J}(\mathrm{H},\mathrm{l})}\left\|\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}\right\|_{\mathrm{L}^{2}(\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}})}^{2} + 2\sum_{\mathrm{j}\neq\mathrm{j}'\text{and }\mathrm{j},\mathrm{j}'\in\mathrm{J}(\mathrm{H},\mathrm{l})}\left\langle\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}},\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}'}\right\rangle_{\mathrm{L}^{2}(\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}})}.
\end{aligned}
$$

Since $\mathrm{J}\left(\mathrm{H},\mathrm{l}\right)$ is a finite set, appealing to the fact that each $\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}$ is square-integrable over a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$ we obtain the desired result

$$
\left\|[\mathrm{J}_{\mathbf{A}}\mathrm{H}]\right\|_{\mathrm{l}^{2}(\Gamma)}^{2} \leq \underbrace{\sum_{\mathrm{l}\in\mathrm{F}}\sum_{\mathrm{j}\in\mathrm{J}(\mathrm{H},\mathrm{l})}\left\|\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}}\right\|_{\mathrm{L}^{2}(\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}})}^{2}}_{\text{finite sum}} + \underbrace{\sum_{\mathrm{l}\in\mathrm{F}}\sum_{\mathrm{j}\neq\mathrm{j}'\text{and }\mathrm{j},\mathrm{j}'\in\mathrm{J}(\mathrm{H},\mathrm{l})}2\left\langle\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}},\Theta_{\mathrm{H},\mathrm{l},\mathrm{j}'}\right\rangle_{\mathrm{L}^{2}(\mathbb{R}^{\mathrm{p}}/\mathbb{Z}^{\mathrm{p}})}}_{\text{finite sum}}.
$$

Thus $\left\|[\mathrm{J}_{\mathbf{A}}\mathrm{H}]\right\|_{\mathrm{l}^{2}(\Gamma)}^{2}$ is finite. $\square$

Let τ be the unitary representation of Γ which acts on the Hilbert space $\mathrm{L}^{2}\left(\mathbf{A}\times\mathbb{R}^{\mathrm{m}},\mathrm{d}\mu\left(\lambda\right)\right)$ as follows:

$$
\left[\tau\left(\gamma\right)\mathrm{F}\right]\left(\lambda,\mathrm{t}\right) = \sigma_{\lambda}\left(\gamma\right)\mathrm{F}\left(\lambda,\mathrm{t}\right).
$$

We shall prove the following three important facts.

- $\mathrm{J}_{\mathbf{A}}$ intertwines τ with the right regular representation of the discrete uniform group Γ (Lemma 25)
- If $\left|\beta\left(\mathbf{A}\times[0,1)^{\mathrm{m}}\right)\right| > 0$ and if $\beta\left(\mathbf{A}\times[0,1)^{\mathrm{m}}\right)$ is contained in a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$ then $\mathrm{J}_{\mathbf{A}}$ defines an isometry on a dense subset of $\mathrm{L}^{2}\left(\mathbf{A}\times\mathbb{R}^{\mathrm{m}},\mathrm{d}\mu\left(\lambda\right)\right)$ into $\mathrm{l}^{2}\left(\Gamma\right)$ which extends uniquely to an isometry of $\mathrm{L}^{2}\left(\mathbf{A}\times\mathbb{R}^{\mathrm{m}},\mathrm{d}\mu\left(\lambda\right)\right)$ into $\mathrm{l}^{2}\left(\Gamma\right)$ (Lemma 26)
- If $\left|\beta\left(\mathbf{A}\times[0,1)^{\mathrm{m}}\right)\right| > 0$ and if $\beta\left(\mathbf{A}\times[0,1)^{\mathrm{m}}\right)$ is up to a null set equal to a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$ then $\mathrm{J}_{\mathbf{A}}$ defines a unitary map (Lemma 27)

Lemma 25. *The map $\mathrm{J}_{\mathbf{A}}$ intertwines τ with the right regular representation of Γ.*

Proof. Let $\mathrm{F}\in\mathrm{L}^{2}\left(\mathbf{A}\times\mathbb{R}^{\mathrm{m}},\mathrm{d}\mu\left(\lambda\right)\right)$ such that F is smooth with compact support. Put

$$
\mathrm{A}\left(\mathrm{l}\right) = \mathrm{l}_{1}\mathrm{A}_{1} + \cdots + \mathrm{l}_{\mathrm{m}}\mathrm{A}_{\mathrm{m}} \text{ and } \mathrm{Z}\left(\mathrm{k}\right) = \mathrm{k}_{1}\mathrm{Z}_{1} + \cdots + \mathrm{k}_{\mathrm{p}}\mathrm{Z}_{\mathrm{p}}
$$

$$
\mathrm{A}\left(\mathrm{l}'\right) = \mathrm{l}_{1}'\mathrm{A}_{1} + \cdots + \mathrm{l}_{\mathrm{m}}'\mathrm{A}_{\mathrm{m}} \text{ and } \mathrm{Z}\left(\mathrm{k}'\right) = \mathrm{k}_{1}'\mathrm{Z}_{1} + \cdots + \mathrm{k}_{\mathrm{p}}'\mathrm{Z}_{\mathrm{p}}.
$$

Firstly,

$$\left[J_{\mathbf{A}} \tau \left(\exp A \left(l' \right) \right) F \right] \left(\exp Z \left(k \right) \exp A \left(l \right) \right)$$

$$= \int_{\mathbf{A}} \int_{[0,1)^m} \left[\sigma_\lambda \left(\exp A \left(l' \right) \right) F \right] \left(\lambda, t - l \right) e^{2\pi i \langle \beta(\lambda,t), Z(k) \rangle} dt d\mu \left(\lambda \right)$$

$$= \int_{\mathbf{A}} \int_{[0,1)^m} F \left(\lambda, t - \left(l + l' \right) \right) e^{2\pi i \langle \beta(\lambda,t), Z(k) \rangle} dt d\mu \left(\lambda \right)$$

$$= J_{\mathbf{A}} F \left(\left(\exp Z \left(k \right) \exp A \left(l + l' \right) \right) \right)$$

$$= R \left(\exp A \left(l' \right) \right) J_{\mathbf{A}} F \left(\exp Z \left(k \right) \exp A \left(l \right) \right).$$

Secondly,

$$\left[J_{\mathbf{A}} \tau \left(\exp Z \left(k' \right) \right) F \right] \left(\exp Z \left(k \right) \exp A \left(l \right) \right)$$

$$= \int_{\mathbf{A}} \int_{[0,1)^m} \left[\sigma_\lambda \left(\exp Z \left(k' \right) \right) F \right] \left(\lambda, t - l \right) e^{2\pi i \langle \beta(\lambda,t), Z(k) \rangle} dt d\mu \left(\lambda \right)$$

$$= \int_{\mathbf{A}} \int_{[0,1)^m} e^{2\pi i \left\langle \lambda, e^{-\mathrm{ad}(t_1 - l_1) A_1 + \cdots + (t_m - l_m) A_m} Z' \right\rangle} F \left(\lambda, t - l \right) e^{2\pi i \langle \beta(\lambda,t), Z(k) \rangle} dt d\mu \left(\lambda \right).$$

Finally,

$$\left[J_{\mathbf{A}} \tau \left(\exp Z \left(k' \right) \right) F \right] \left(\exp Z \left(k \right) \exp A \left(l \right) \right) = \int_{\mathbf{A}} \int_{[0,1)^m} e^{2\pi i \langle \beta(\lambda,t), Z(k) + e^A Z(k') \rangle} F \left(\lambda, t - l \right) dt d\mu \left(\lambda \right)$$

$$= J F \left(\exp \left(Z \left(k \right) + e^{\mathrm{ad} A} Z \left(k' \right) \right) \exp A \left(l \right) \right)$$

$$= \left[R \left(\exp Z \left(k' \right) \right) J F \right] \left(\exp Z \left(k \right) \exp A \left(l \right) \right).$$

This completes the proof. □

Lemma 26. *If $\beta \left(\mathbf{A} \times [0,1)^m \right)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$ then $J_{\mathbf{A}}$ defines an isometry on a dense subset of $L^2 \left(\mathbf{A} \times \mathbb{R}^m, d\mu \left(\lambda \right) \right)$ into $l^2 \left(\Gamma \right)$ which extends uniquely to an isometry of $L^2 \left(\mathbf{A} \times \mathbb{R}^m, d\mu \left(\lambda \right) \right)$ into $l^2 \left(\Gamma \right).$*

Proof. Let $F \in L^2 \left(\mathbf{A} \times \mathbb{R}^m, d\mu \left(\lambda \right) \right).$ Furthermore, let us assume that F is smooth with compact support in $\mathbf{A} \times \mathbb{R}^m$. Computing the norm of F, we obtain

$$\|F\|^2_{L^2(\mathbf{A} \times \mathbb{R}^m, d\mu(\lambda))} = \int_{\mathbf{A}} \int_{\mathbb{R}^m} |F \left(\lambda, t \right)|^2 dt \, d\mu \left(\lambda \right)$$

(30)

$$= \int_{\mathbf{A}} \sum_{l \in \mathbb{Z}^m} \int_{[0,1)^m} |F \left(\lambda, t - l \right)|^2 dt \, d\mu \left(\lambda \right).$$

Letting $G_l(\lambda, t) = F(\lambda, t-1)$, making the change of variable $s = \beta(\lambda, t)$ and using the fact that β is a diffeomorphism (see Lemma 19) we obtain

$$\int_{\mathbf{A}} \sum_{l \in \mathbb{Z}^m} \int_{[0,1)^m} |F(\lambda, t-1)|^2 \, dt \, d\mu(\lambda) = \sum_{l \in \mathbb{Z}^m} \int_{[0,1)^m} \int_{\mathbf{A}} |G_l(\lambda, t)|^2 \, d\mu(\lambda) \, dt$$

$$= \sum_{l \in \mathbb{Z}^m} \int_{\mathbf{A} \times [0,1)^m} |G_l(\lambda, t)|^2 \left| \det J_\beta(\lambda, 0) \right| d(\lambda, t)$$

$$= \sum_{l \in \mathbb{Z}^m} \int_{\beta(\mathbf{A} \times [0,1)^m)} \left| G_l \beta^{-1}(s) \right|^2 \left| \det J_\beta(\lambda, 0) \right| d\left(\beta^{-1}(s) \right)$$

$$= \sum_{l \in \mathbb{Z}^m} \int_{\beta(\mathbf{A} \times [0,1)^m)} \left| G_l \beta^{-1}(s) \right|^2 ds.$$

Since $\beta(\mathbf{A} \times [0,1)^m)$ is contained in a fundamental domain of $\mathbb{Z}^p$ then

$$\int_{\mathbf{A}} \sum_{l \in \mathbb{Z}^m} \int_{[0,1)^m} |F(\lambda, t-1)|^2 \, dt \, d\mu(\lambda) = \sum_{l \in \mathbb{Z}^m} \left\| s \mapsto G_l \beta^{-1}(s) \right\|^2$$

$$= \sum_{l \in \mathbb{Z}^m} \left\| \mathfrak{F}_{\mathbb{R}^p / \mathbb{Z}^p} \left(G_l \beta^{-1} \right) \right\|^2$$

$$= \sum_{l \in \mathbb{Z}^m} \sum_{k \in \mathbb{Z}^p} \left| \mathfrak{F}_{\mathbb{R}^p / \mathbb{Z}^p} \left(G_l \beta^{-1} \right)(k) \right|^2$$

$$= \sum_{l \in \mathbb{Z}^m} \sum_{k \in \mathbb{Z}^p} \left| \int_{\beta(\mathbf{A} \times [0,1)^m)} \left(G_l \beta^{-1} \right)(s) \exp(2\pi i \langle s, k \rangle) \, ds \right|^2.$$

Thus

$$(31) \qquad \|F\|^2_{L^2(\mathbf{A} \times \mathbb{R}^m, d\mu(\lambda))} = \sum_{l \in \mathbb{Z}^m} \sum_{k \in \mathbb{Z}^p} \left| \int_{\beta(\mathbf{A} \times [0,1)^m)} G_l \beta^{-1}(s) \exp(2\pi i \langle s, k \rangle) \, ds \right|^2.$$

Next, since $\beta(\mathbf{A} \times [0,1)^m)$ is contained in a fundamental domain of $\mathbb{Z}^p$, the trigonometric system

$$\left\{ \chi_{\beta(\mathbf{A} \times [0,1)^m)}(s) \times \exp(2\pi i \langle s, k \rangle) : k \in \mathbb{Z}^p \right\}$$

forms a Parseval frame in $L^2(\beta(\mathbf{A} \times [0,1)^m))$. Clearly this is true because the orthogonal projection of an orthonormal basis is always a Parseval frame. Letting

$$\widehat{G_l \beta^{-1}} = \mathfrak{F}_{L^2(\beta(\mathbf{A} \times [0,1)^m))} \left(s \mapsto G_l \left(\beta^{-1}(s) \right) \right)$$

be the Fourier transform of the function

$$s \mapsto G_l \left(\beta^{-1}(s) \right) \in L^2(\beta(\mathbf{A} \times [0,1)^m))$$

it follows that

$$\sum_{l \in \mathbb{Z}^m} \sum_{k \in \mathbb{Z}^p} \left| \int_{\beta(\mathbf{A} \times [0,1)^m)} G_l \left(\beta^{-1}(s) \right) \exp(2\pi i \langle s, k \rangle) \, ds \right|^2 = \sum_{l \in \mathbb{Z}^m} \sum_{k \in \mathbb{Z}^p} \left| \widehat{G_l \beta^{-1}}(k) \right|^2.$$

Next,

$$\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\widehat{G_l\beta^{-1}}\left(k\right)\right|^2 = \sum_{l\in\mathbb{Z}^m}\left\|\widehat{G_l\beta^{-1}}\right\|^2_{l^2(\mathbb{Z}^p)} = \sum_{l\in\mathbb{Z}^m}\int_{\beta(\mathbf{A}\times[0,1)^m)}\left|G_l\left(\beta^{-1}\left(s\right)\right)\right|^2 ds.$$

Now substituting (λ, t) for $\beta^{-1}(s)$,

$$\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\int_{\beta(\mathbf{A}\times[0,1)^m)}G_l\left(\beta^{-1}\left(s\right)\right)\exp\left(2\pi i\left\langle s, k\right\rangle\right)ds\right|^2$$

(32)
$$= \sum_{l\in\mathbb{Z}^m}\int_{\mathbf{A}}\int_{[0,1)^m}\left|G_l\left(\lambda, t\right)\right|^2\left|\det J_\beta\left(\lambda, 0\right)\right|dt\ d\lambda$$

(33)
$$= \sum_{l\in\mathbb{Z}^m}\int_{\mathbf{A}}\int_{[0,1)^m}\left|F\left(\lambda, t-l\right)\right|^2 dt\ d\mu\left(\lambda\right).$$

Equation (30) together with (33) gives

$$\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\int_{\beta(\mathbf{A}\times[0,1)^m)}G_l\left(\beta^{-1}\left(s\right)\right)\exp\left(2\pi i\left\langle s, k\right\rangle\right)\ ds\right|^2 = \|F\|^2_{L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda))}.$$

Finally, we obtain

$$\|F\|_{L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda))} = \|J_{\mathbf{A}}F\|_{l^2(\Gamma)}.$$

Now, since the set of continuous functions of compact support is dense in $L^2\left(\mathbf{A}\times\mathbb{R}^m, d\mu\left(\lambda\right)\right)$ and since J defines an isometry on a dense set, then J extends uniquely to an isometry on $L^2\left(\mathbf{A}\times\mathbb{R}^m, d\mu\left(\lambda\right)\right).$ $\square$

The proof given for Lemma 26 can be easily modified to establish the following result

Lemma 27. *If $\beta\left(\mathbf{A}\times[0,1)^m\right)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is equal to a fundamental domain of $\mathbb{Z}^p$ then $J_{\mathbf{A}}$ defines an isometry on a dense subset of $L^2\left(\mathbf{A}\times\mathbb{R}^m, d\mu\left(\lambda\right)\right)$ into $l^2\left(\Gamma\right)$ which extends uniquely to a unitary map of $L^2\left(\mathbf{A}\times\mathbb{R}^m, d\mu\left(\lambda\right)\right)$ into $l^2\left(\Gamma\right).$*

Remark 28. *Suppose that $\beta\left(\mathbf{A}\times[0,1)^m\right)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$. We have shown that $J_{\mathbf{A}}$ is an isometry. Now, let Φ be the orthogonal projection of $l^2\left(\Gamma\right)$ onto the Hilbert space $J_{\mathbf{A}}\left(L^2\left(\mathbf{A}\times\mathbb{R}^m, d\mu\left(\lambda\right)\right)\right)$ and let κ be the indicator sequence of the singleton containing the identity element in Γ. Identifying $L^2\left(\mathbf{A}\times\mathbb{R}^m, d\mu\left(\lambda\right)\right)$ with $\mathcal{P}\left(\mathbf{H_A}\right)$, it is clear that $\mathcal{P}^{-1}\left(J_{\mathbf{A}}^*\left(\Phi\kappa\right)\right)\in\mathbf{H_A}\subset L^2\left(\mathbb{N}\right)$ and $L\left(\Gamma\right)\left(\mathcal{P}^{-1}\left(J_{\mathbf{A}}^*\left(\Phi\kappa\right)\right)\right)$ is a Parseval frame for the band-limited Hilbert space $\mathbf{H_A}$. We remark that the vector κ could be replaced by any other vector which generates an orthonormal basis or a Parseval frame under the action of the right regular representation of Γ.*

4. Proof of Main Results

4.1. **Proof of Theorem 4.** First, we observe that the right regular and left regular representations of Γ are unitarily equivalent ([7], Page 69). To prove Part 1, we appeal to Lemma 26, and Lemma 25. Assuming that $\beta\left(\mathbf{A}\times[0,1)^m\right)$ has positive Lebesgue

measure in $\mathbb{R}^{\mathrm{p}}$ and is contained in a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$, the restriction of the representation $(\mathrm{L}, \mathbf{H_A})$ to the discrete group Γ is equivalent to a subrepresentation of the left regular representation of Γ. Appealing to Lemma 11, there exists a vector η such that $\mathrm{V}_\eta (\mathbf{H_A})$ is a sampling space with respect to Γ. In fact, Remark 28 describes how to construct η. For Part 2, Lemma 26, Lemma 25 together with the assumption that $\beta (\mathbf{A} \times [0,1)^{\mathrm{m}})$ is equal to a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$ imply that the restriction of the representation $(\mathrm{L}, \mathbf{H_A})$ to the discrete group Γ is equivalent to the left regular representation of Γ. Finally, Remark 28 shows how to construct a vector $\eta \in \mathbf{H_A}$ such that $\mathrm{V}_\eta^{\mathrm{L}} (\mathbf{H_A})$ is a left-invariant subspace of $\mathrm{L}^2 (\mathrm{N})$ which is a sampling space with the interpolation property with respect to Γ. This completes the proof.

4.2. **Proof of Corollary 5.** For $\mathrm{s} = (\mathrm{s}_1, \cdots, \mathrm{s_m}) \in \mathbb{R}^{\mathrm{m}}$, let $\mathrm{A}(\mathrm{s}) = \mathrm{s}_1 \mathrm{A}_1 + \cdots + \mathrm{s_m} \mathrm{A_m} \in \mathfrak{m}$. Since the linear operators $\mathrm{adA}_1, \cdots, \mathrm{adA_m}$ are pairwise commutative and nilpotent, since $\mathrm{e}^{-\mathrm{adA(s)}|\mathfrak{p}}$ is unipotent, there is a unit vector which is an eigenvector for $\mathrm{e}^{-\mathrm{adA(s)}|\mathfrak{p}}$ with corresponding eigenvalue 1. So, it is clear that $\left\| \left[\mathrm{e}^{-\mathrm{adA(s)}|\mathfrak{p}} \right]^{\mathrm{T}} \right\|_\infty \geq 1$ and

$$\sup \left\{ \left\| \left[\mathrm{e}^{-\mathrm{adA(s)}|\mathfrak{p}} \right]^{\mathrm{T}} \right\|_\infty : \mathrm{s} \in \mathbf{E} \right\} \geq 1$$

for any nonempty $\mathbf{E} \subseteq \mathbb{R}^{\mathrm{m}}$. We recall again that

$$(34) \qquad \mathfrak{P}(\mathrm{A}(\mathrm{s})) = \left[\mathrm{e}^{\mathrm{ad}\left(-\sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{s_j A_j} \right)|\mathfrak{p}} \right]^{\mathrm{T}}.$$

Lemma 29. *Let $\mathbf{E}$ be an open bounded subset of $\mathbb{R}^{\mathrm{m}}$. If ε is a positive number satisfying*

$$\varepsilon \leq \delta = (2 \sup \{ \| \mathfrak{P}(\mathrm{A}(\mathrm{s})) \|_\infty : \mathrm{s} \in \mathbf{E} \})^{-1}$$

then $\beta \left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma \right) \times \mathbf{E} \right)$ is open in $\mathbb{R}^{\mathrm{p}}$ and is contained in a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$.

Proof. Since the map β is a diffeomorphism (see Lemma 19) and since the set

$$\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma \right) \times \mathbf{E}$$

is an open set in $\Sigma \times \mathbb{R}^{\mathrm{m}}$, it is clear that its image under the map β is also open in $\mathbb{R}^{\mathrm{p}}$. Next, it remains to show that it is possible to find a positive real number δ such that if $0 < \varepsilon \leq \delta$ then $\beta \left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma \right) \times \mathbf{E} \right)$ is an open set contained in a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$. Let $\lambda \in \Sigma$. Then there exists a linear functional f in the dual of the ideal $\mathfrak{p}$ such that

$$\iota(\lambda) = \left[\begin{array}{c} \mathrm{f} \\ 0 \end{array} \right]$$

and

$$(35) \qquad \iota \left(\exp \left(\sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{s_j A_j} \right) \cdot \lambda \right) = \left[\begin{array}{c} \mathfrak{P}(\mathrm{A}(\mathrm{s})) \, \mathrm{f} \\ 0 \end{array} \right].$$

Moreover, it is worth noting that

$$\left\| \exp\left(\sum_{j=1}^{m} s_j A_j \right) \cdot \lambda \right\|_{\max} = \left\| \mathfrak{P}\left(A(s) \right) f \right\|_{\max}.$$

Let δ be a positive real number defined as follows:

(36) $$\delta = \left(2 \sup \left\{ \left\| \mathfrak{P}\left(A(s) \right) \right\|_{\infty} : s \in \mathbf{E} \right\} \right)^{-1}.$$

If $f \in (-\varepsilon, \varepsilon)^{\dim \Sigma} \subseteq (-\delta, \delta)^{\dim \Sigma}$ and if $s \in \mathbf{E}$ then

$$\left\| \mathfrak{B}(A(s)) f \right\|_{\infty} \leq \left\| f \right\|_{\max} \times \sup \left\{ \left\| \mathfrak{P}\left(A(s) \right) \right\|_{\infty} : s \in \mathbf{E} \right\}$$

$$= \frac{1}{2} \times \frac{\left\| f \right\|_{\max}}{\delta}.$$

Now since $\left\| f \right\|_{\max} < \delta$, it follows that

$$\left\| \mathfrak{P}\left(A(s) \right) f \right\|_{\max} < \frac{1}{2}.$$

As a result,

$$\beta \left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma \right) \times \mathbf{E} \right) \subseteq \left(-\frac{1}{2}, \frac{1}{2} \right)^{p}$$

and clearly $\left(-\frac{1}{2}, \frac{1}{2} \right)^{p}$ is contained in a fundamental domain of $\mathbb{Z}^p$. □

Appealing to Lemma 26, and Lemma 29 the following is immediate

Proposition 30. *If*

$$0 < \varepsilon \leq \delta = \frac{1}{2 \sup \left\{ \left\| \mathfrak{P}\left(A(s) \right) \right\|_{\infty} : s \in \mathbf{E} \right\}}$$

then $J_{(-\varepsilon, \varepsilon)^{n-2m} \cap \Sigma}$ *defines an isometry between* $L^2 \left(\left((-\varepsilon, \varepsilon)^{n-2m} \cap \Sigma \right) \times \mathbb{R}^m, d\mu(\lambda) \right)$ *and* $l^2(\Gamma)$.

4.2.1. *Proof of Corollary 5.* Let δ be a positive number defined by

(37) $$\delta = \frac{1}{2 \sup \left\{ \left\| \mathfrak{P}\left(A(s) \right) \right\|_{\infty} : s \in [0,1)^m \right\}}.$$

We want to show that for $\varepsilon \in (0, \delta]$ there exists a band-limited vector

$$\eta = \eta^{\varepsilon} \in \mathbf{H}_{(-\varepsilon, \varepsilon)^{n-2m}}$$

such that the Hilbert space $V_{\eta}^{L} \left(\mathbf{H}_{(-\varepsilon, \varepsilon)^{n-2m}} \right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ. According to Lemma 29 the set

$$\beta \left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma \right) \times [0,1)^m \right)$$

is open in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$. The desired result follows immediately from Theorem 4.

4.3. **Proof of Example 6 Part** 1. The case of commutative simply connected, and connected nilpotent Lie group is already known to be true. Thus, to prove this result, it remains to focus on the non commutative algebras. According to the classification of four-dimensional nilpotent Lie algebras [11] there are three distinct cases to consider. Indeed if $\mathfrak{n}$ is a non-commutative nilpotent Lie algebra of dimension three, then $\mathfrak{n}$ must be isomorphic with the three-dimensional Heisenberg Lie algebra. If $\mathfrak{n}$ is four-dimensional then up to isomorphism either $\mathfrak{n}$ is the direct sum of the Heisenberg Lie algebra with a one-dimensional algebra, or there is a basis Z_1, Z_2, Z_3, A_1 for $\mathfrak{n}$ with the following non-trivial Lie brackets

$$[A_1, Z_2] = 2Z_1, [A_1, Z_3] = 2Z_2.$$

Case 1 (The Heisenberg Lie algebra) Let N be the simply connected, connected Heisenberg Lie group with Lie algebra $\mathfrak{n}$ which is spanned by Z_1, Z_2, A_1 with non-trivial Lie brackets $[A_1, Z_2] = Z_1$. We check that $N = PM$ where $P = \exp(\mathbb{R}Z_1 + \mathbb{R}Z_2)$ and $M = \exp(\mathbb{R}A_1)$. Put

$$\Gamma = \exp(\mathbb{Z}Z_1)\exp(\mathbb{Z}Z_2)\exp(\mathbb{Z}A_1).$$

It is easily checked that

$$\mathbf{M}(\lambda) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda(Z_1) \\ 0 & \lambda(Z_1) & 0 \end{bmatrix}.$$

Next, since

$$\mathbf{e}(\lambda) = \begin{cases} \emptyset \text{ if } \lambda(Z_1) = 0 \\ \{2,3\} \text{ if } \lambda(Z_1) \neq 0 \end{cases}$$

we obtain that $\mathbf{e} = \{2,3\}$. It follows that $\Omega_{\mathbf{e}} = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1) \neq 0\}$. Next, the unitary dual of N is parametrized by $\Sigma = \{\lambda \in \Omega_{\mathbf{e}} : \lambda(Z_2) = \lambda(A_1) = 0\}$ which we identify with the punctured line: $\mathbb{R}^*$. It is not hard to check that
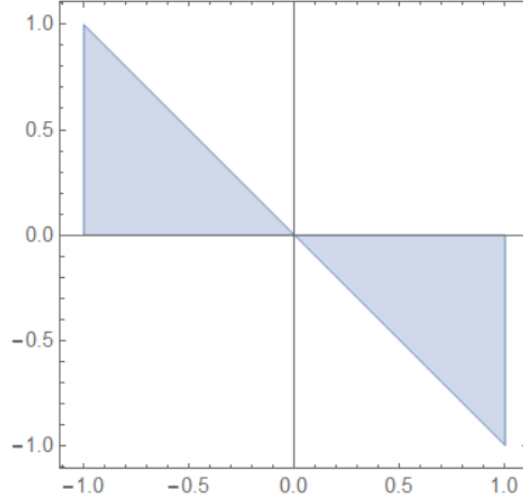
$$\delta^{-1} = 2\sup\left\{ \left\| \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} = 4.$$

So, there exists a band-limited vector $\eta \in \mathbf{H}_{\left(-\frac{1}{4}, \frac{1}{4}\right)}$ such that $V_\eta^L\left(\mathbf{H}_{\left(-\frac{1}{4}, \frac{1}{4}\right)}\right)$ is a sampling space with respect to Γ.

To prove that the Heisenberg group admits sampling spaces with the interpolation property with respect to Γ, we claim that the set

$$B(1) = \beta\left((-1,1) \times [0,1)\right) = \left\{ \begin{bmatrix} f \\ -sf \end{bmatrix} : f \in (-1,1), s \in [0,1) \right\}$$

is up to a null set equal to a fundamental domain of $\mathbb{Z}^2$ (see illustration below)

The set B(1)

To prove this we write

$$\beta\left((-1,1) \times [0,1)\right) = \beta\left((0,1) \times [0,1)\right) \cup \beta\left((-1,0) \times [0,1)\right).$$

Next, it is easy to check that

$$\left(\beta\left((0,1) \times [0,1)\right) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \cup \left(\beta\left((-1,0) \times [0,1)\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

is up to a null set equal to the unit square $[0,1)^2$. Thus the set $\beta\left((-1,1) \times [0,1)\right)$ is up to a null set equal to a fundamental domain of $\mathbb{Z}^2$. Appealing to Theorem 4, the following result confirms the work proved in [6][17]. There exists a band-limited vector $\eta \in \mathbf{H}_{(-1,1)}$ such that $V_\eta^{\mathrm{L}}\left(\mathbf{H}_{(-1,1)}\right)$ is a sampling space with respect to Γ which also enjoys the **interpolation property**.

Case 2 (Four-dimensional and step two) Assume that $\mathfrak{n}$ is the direct sum of the Heisenberg Lie algebra with $\mathbb{R}$. That is $\mathfrak{n}$ which is spanned by Z_1, Z_2, Z_3, A_1 with non-trivial Lie brackets $[A_1, Z_2] = Z_1$. We check that

$$\mathbf{M}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(Z_1) \\ 0 & 0 & 0 & 0 \\ 0 & \lambda(Z_1) & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{e}(\lambda) = \begin{cases} \emptyset \text{ if } \lambda(Z_1) = 0 \\ \{2,4\} \text{ if } \lambda(Z_1) \neq 0 \end{cases}.$$

Fix $\mathbf{e} = \{2,4\}$ such that $\Omega_{\mathbf{e}} = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1) \neq 0\}$ and the unitary dual of N is parametrized by

$$\Sigma = \{\lambda \in \Omega_{\mathbf{e}} : \lambda(Z_2) = \lambda(A_1) = 0\}.$$

For any linear functional $\lambda \in \Sigma$, the ideal spanned by Z_1, Z_2, Z_3 is a polarization algebra subordinated to λ and

$$\delta = \left(2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} \right)^{-1} = \frac{1}{4}.$$

Case 3 (Four-dimensional and three step) Assume that $\mathfrak{n}$ is a four-dimensional Z_1, Z_2, Z_3, A_1 such that

$$[A_1, Z_2] = 2Z_1, [A_1, Z_3] = 2Z_2.$$

With respect to the ordered basis Z_1, Z_2, Z_3, we have

$$[\text{ad}A_1]|\, \mathfrak{p} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \exp [\text{ad}A_1]|\, \mathfrak{p} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, we check that

$$\delta = \left(2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 & 0 \\ -2s & 1 & 0 \\ 2s^2 & -2s & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} \right)^{-1}$$

$$= \frac{1}{2} \left(\max \left\{ 1, 1 + 2\,|s|, 1 + 2\,|s| + 2\,|s|^2 : s \in [0,1) \right\} \right)^{-1} = \frac{1}{10}.$$

Indeed, the set

$$\beta \left(\left(-\frac{1}{10}, \frac{1}{10} \right)^2 \times [0,1) \right) = \left\{ \begin{bmatrix} \lambda_1 \\ -2s\lambda_1 \\ 2\lambda_1 s^2 + \lambda_2 \end{bmatrix} : (\lambda_1, \lambda_1, s) \in \left(-\frac{1}{10}, \frac{1}{10} \right)^2 \times [0,1) \right\}$$

$$\subset \left(-\frac{1}{2}, \frac{1}{2} \right)^3$$

is contained in a fundamental domain of $\mathbb{Z}^3$. Thus, there exists a band-limited vector $\eta \in \mathbf{H}_{\left(-\frac{1}{10}, \frac{1}{10}\right)}$ such that $V_\eta^L \left(\mathbf{H}_{\left(-\frac{1}{10}, \frac{1}{10}\right)} \right)$ is a sampling space with respect to

$$\Gamma = \exp \left(\mathbb{Z}Z_1 + \mathbb{Z}Z_2 + \mathbb{Z}Z_3 \right) \exp \left(\mathbb{Z}A_1 \right).$$

4.4. **Proof of Example 6 Part** 2 **.** Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, \cdots, Z_p, A_1$ such that $[\text{ad}A_1]|_\mathfrak{p} = A$ is a nonzero rational upper triangular nilpotent matrix of order p such that $e^A \mathbb{Z}^p \subseteq \mathbb{Z}^p$ and the algebra generated by $Z_1, Z_2, \cdots, Z_p$ is commutative. Then N is isomorphic to a semi-direct product group $\mathbb{R}^p \rtimes \mathbb{R}$ with multiplication law given by

$$(x, t) \left(x', t' \right) = \left(x + e^{tA}x', t + t' \right).$$

Clearly since A is not the zero matrix then

$$\max \left\{ \text{rank} \left(\mathbf{M} \left(\lambda \right) \right) : \lambda \in \mathfrak{n}^* \right\} = 2$$

and the unitary dual of N is parametrized by a Zariski open subset of $\mathbb{R}^{p-1}$. Finally, let

$$\delta = \left(2 \times \sup\left\{\left\|\sum_{k=0}^{m-1} \frac{\left(-sA^{T}\right)^{k}}{k!}\right\|_{\infty} : s \in [0,1)\right\}\right)^{-1} > 0.$$

For $\varepsilon \in (0, \delta]$ there exists a band-limited vector $\eta = \eta^{\varepsilon} \in \mathbf{H}_{(-\varepsilon,\varepsilon)^{p-1}}$ such that the Hilbert space $V_{\eta}^{L}\left(\mathbf{H}_{(-\varepsilon,\varepsilon)^{p-1}}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.

4.5. **Proof of Example 6 Part** 3. Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, \cdots, Z_p, A_1, \cdots, A_m$ where $p = m+1$ and the matrix representation of $\mathrm{ad}\left(\sum_{k=1}^{m} t_k A_k\right)$ restricted to $\mathfrak{p}$ is given by the following matrix of order $m + 1$

$$A(t) = \left[\mathrm{ad}\sum_{k=1}^{m} t_k A_k\right]\Big|\mathfrak{p} = m! \begin{bmatrix} 0 & t_1 & t_2 & \cdots & t_{m-1} & t_m \\ & 0 & t_1 & t_2 & \ddots & t_{m-1} \\ & & 0 & t_1 & \ddots & \vdots \\ & & & 0 & \ddots & t_2 \\ & & & & \ddots & t_1 \\ & & & & & 0 \end{bmatrix}.$$

We observe that

$$\exp A(t) = \sum_{k=0}^{m+1} \frac{A(t)^{k}}{k!}.$$

Therefore, N is a nilpotent Lie group of step $p = m + 1$. Moreover, the unitary dual of N is parametrized by the manifold:

$$\Sigma = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1) \neq 0$$
$$\text{and } \lambda(Z_{k+1}) = \lambda(A_k) = 0 \text{ for } 1 \leq k \leq m\} \simeq \mathbb{R}^*$$

and the Plancherel measure is up to multiplication by a constant given by $|\lambda|^m \, d\lambda$. Let

$$r(t) = 2\left\|\sum_{k=0}^{m+1} \frac{\left(-A(t)^{T}\right)^{k}}{k!}\right\|_{\infty}$$

be a function defined on $\mathbb{R}^m$. The positive number δ described in Corollary 5 is equal to

$$\delta = \left(\sup\left\{r(t) : t \in [0,1)^{m}\right\}\right)^{-1}.$$

Thus, for $\varepsilon \in (0, \delta]$ there exists a band-limited vector $\eta = \eta^{\varepsilon} \in \mathbf{H}_{(-\varepsilon,\varepsilon)}$ such that the Hilbert space $V_{\eta}^{L}\left(\mathbf{H}_{(-\varepsilon,\varepsilon)}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.

5. Construction of Other Sampling Sets

In this section, we shall describe how to construct other sampling sets for band-limited multiplicity-free spaces from a fixed given sampling set Γ. Assume that α is an automorphism of the Lie group N. Then α induces the following unitary map $D : L^2(N) \to L^2(N)$ which is defined as follows: $[Dh](n) = \Lambda(\alpha)^{-1/2} h(\alpha^{-1}(n))$ where

$$\Lambda(\alpha) = \frac{d(\alpha(n))}{dn}.$$

Lemma 31. *For any* $x \in N$, $DL(x)D^{-1} = L(\alpha(x))$.

Proof. Let $h \in L^2(N)$. Then $\left[DL(x)D^{-1}h\right](n) = \Lambda(\alpha)^{-1/2}\left[D^{-1}h\right]\left(x^{-1}\alpha^{-1}(n)\right)$ and

$$\left[DL(x)D^{-1}h\right](n) = h\left(\alpha\left(x^{-1}\right)\alpha\left(\alpha^{-1}(n)\right)\right)$$
$$= h\left(\alpha\left(x^{-1}\right)n\right) = [L(\alpha(x))h](n).$$

$\square$

Lemma 32. *Let* $\mathbf{H_A}$ *be as defined in (15). The image of the Hilbert space* $\mathbf{H_A}$ *under the unitary map* D *is band-limited and is multiplicity-free.*

Proof. It is well-known that (see Proposition 1.2, [12]) for $n \in N$,

$$\mathrm{Ind}_P^N(\chi_\lambda) \circ \alpha(n) = C \circ \left[\mathrm{Ind}_{\alpha^{-1}(P)}^N(\chi_\lambda \circ \alpha)(n)\right] \circ C^*$$

for some unitary operator C acting $L^2(\mathbb{R}^m)$ which is unique up to multiplication by a complex number of magnitude one (according to Schur's lemma). Now, let H be the group generated by the automorphism α. Let $\left[\mathrm{Ind}_P^N(\chi_\lambda) \circ \alpha^{-1}\right]$ be the class of irreducible representations of N which are equivalent to

$$\mathrm{Ind}_P^N(\chi_\lambda) \circ \alpha^{-1}.$$

Then H acts on the unitary dual of N as follows

$$\alpha \star \lambda = \left[\mathrm{Ind}_P^N(\chi_\lambda) \circ \alpha^{-1}\right].$$

Next, let $h \in \mathbf{H_A}$ and let $\mathbf{w}, \mathbf{v} \in L^2(\mathbb{R}^m)$. Then

$$\langle[\mathcal{P}Dh](\lambda)\mathbf{w}, \mathbf{v}\rangle := \langle[\mathcal{P}Dh](\sigma_\lambda)\mathbf{w}, \mathbf{v}\rangle$$
$$= \Lambda(\alpha)^{1/2}\int_N h(n)\langle\sigma_\lambda(\alpha(n))\mathbf{w}, \mathbf{v}\rangle dn$$
$$= \Lambda(\alpha)^{1/2}\left\langle\left[C \circ \mathcal{P}h\left(\alpha^{-1} \star \lambda\right) \circ C^*\right]\mathbf{w}, \mathbf{v}\right\rangle.$$

Since

$$\langle[\mathcal{P}Dh](\lambda)\mathbf{w}, \mathbf{v}\rangle = \left\langle\Lambda(\alpha)^{1/2}\left[C \circ \mathcal{P}h\left(\alpha^{-1} \star \lambda\right) \circ C^*\right]\mathbf{w}, \mathbf{v}\right\rangle$$

for arbitrary vectors $\mathbf{w}, \mathbf{v} \in L^2(\mathbb{R}^m)$, it follows that

$$\mathcal{P}(Dh)(\lambda) = \Lambda(\alpha)^{1/2} C \circ (\mathcal{P}h)\left(\alpha^{-1} \star \lambda\right) \circ C^*.$$

Thus, the image of the Hilbert space $\mathbf{H_A}$ under the unitary map D is band-limited and is multiplicity-free. $\qquad\square$

Proposition 33. *If $\beta\left(\mathbf{A}\times[0,1)^{\mathrm{m}}\right)$ is contained in a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$ then there exists a Parseval frame of the type $\{\mathrm{L}\left(\gamma\right)z:\gamma\in\alpha\left(\Gamma\right)\}$ for $\mathrm{D}\mathbf{H_A}$ and there exists a vector $\eta\in\mathrm{D}\mathbf{H_A}$ such that $\mathrm{V}_\eta\left(\mathrm{D}\mathbf{H_A}\right)$ is a sampling space with respect to $\alpha\left(\Gamma\right)$.*

Proof. Let us suppose that $\beta\left(\mathbf{A}\times[0,1)^{\mathrm{m}}\right)$ is contained in a fundamental domain of $\mathbb{Z}^{\mathrm{p}}$. Let $\{\mathrm{L}\left(\gamma\right)\mathrm{h}:\gamma\in\Gamma\}$ be a Parseval frame for $\mathbf{H_A}$. Then $\{\mathrm{D}\left(\mathrm{L}\left(\gamma\right)\mathrm{h}\right):\gamma\in\Gamma\}$ is a Parseval frame for the Hilbert space $\mathrm{D}\mathbf{H_A}$ and since

$$\mathrm{DL}\left(\gamma\right)\mathrm{D}^{-1}=\mathrm{L}\left(\alpha\left(\gamma\right)\right)$$

it follows that

$$\{\mathrm{L}\left(\alpha\left(\gamma\right)\right)\mathrm{Dh}:\gamma\in\Gamma\}=\{\mathrm{L}\left(\gamma\right)\mathrm{Dh}:\gamma\in\alpha\left(\Gamma\right)\}$$

is a Parseval frame for $\mathrm{D}\mathbf{H_A}$. The fact that there exists a vector $\eta\in\mathrm{D}\mathbf{H_A}$ such that $\mathrm{V}_\eta\left(\mathrm{D}\mathbf{H_A}\right)$ is a sampling space with respect to $\alpha\left(\Gamma\right)$ is due to Proposition 2.60 and Proposition 2.54 [9]. $\qquad\square$

6. Concluding Observations

Let us conclude this work by exhibiting an example which does not belong to the class of groups presented described in Condition 3. Let $\mathfrak{n}$ be a five-dimensional nilpotent Lie algebra with basis spanned by Z_1,Z_2,Z_3,A_1,A_2 such that $[A_1,Z_3]=Z_2,[A_2,Z_3]=Z_1$. Next, let $\mathfrak{p}$ be the ideal spanned by Z_1,Z_2,Z_3 and let $\mathfrak{m}$ be the ideal spanned by A_1,A_2. Let N be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Then $N=P\rtimes M$ is a metabelian nilpotent Lie group, and its dual is parametrized by the set

$$\Sigma=\{\zeta_1Z_1^*+\zeta_2Z_2^*+\alpha_1A_1^*:\zeta_1\neq0\}$$

which is a cross-section for all coadjoint orbits in the Zariski open set

$$\Omega=\{\zeta_1Z_1^*+\zeta_2Z_2^*+\zeta_3Z_3^*+\alpha_1A_1^*+\alpha_2A_2^*:\zeta_1\neq0\}.$$

The coadjoint orbits in Ω are two-dimensional manifolds, the ideal $\mathfrak{p}$ is not a polarization for any linear functional λ in Ω. In fact it is properly contained in one. Indeed for any linear functional in the cross-section Σ, a polarization algebra subordinated to λ must be a four-dimensional algebra. For example the set

$$\left\{Z_1,Z_2,Z_3,A_1-\frac{\zeta_2}{\zeta_1}A_2\right\}$$

spans a polarization subordinated to $\zeta\in\Omega$. If there exists a subalgebra of $\mathfrak{n}$ which is a constant polarization, then such an algebra must be four-dimensional. However, there is no four-dimensional subalgebra of $\mathfrak{n}$ which is a maximal commutative ideal. Thus, the results proved in this work do not apply to this group. To the best of our knowledge, it is an open question if the results of this paper extend to nilpotent Lie groups which do not belong to the class of groups considered here. This problem will be the focus of a future investigation.

References

[1] M. Bekka, P. Driutti, Restrictions of irreducible unitary representations of nilpotent Lie groups to lattices. J. Funct. Anal. 168 (1999), no. 2, 514–528.

[2] P.G. Casazza, The Art of Frame Theory, Taiwanese Journal of Math, Vol 4 (2) (2000) 129-202.

[3] L. Corwin, F. Greenleaf, Representations of Nilpotent Lie Groups and their Applications. Part I. Basic Theory and Examples, Cambridge Studies in Advanced Mathematics, 18. Cambridge University Press, Cambridge, (1990).

[4] B. Currey, A. Mayeli, V.Oussa, Shift-invariant spaces on SI/Z Lie groups, Journal of Fourier Analysis and Applications, April 2014, Volume 20, Issue 2, pp 384-400.

[5] B. Currey, Admissibility for a class of quasiregular representations, Canadian Journal of Mathematics, Vol 59, No. 5 (2007) 917-942

[6] B. Currey, A. Mayeli, A Density Condition for Interpolation on the Heisenberg Group, Rocky Mountain J. Math. Volume 42, Number 4 (2012), 1135-1151.

[7] G. Folland, A Course in Abstract Harmonic Analysis, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[8] H. Führ, Admissible vectors for the regular representation Proceedings of the AMS 130 2959–2970 (2002).

[9] H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Springer Lecture Notes in Math. 1863, (2005).

[10] H. Führ, K. Grochenig, Sampling theorems on locally compact groups from oscillation estimates Mathematische Zeitschrift 255, 177-194 (2007).

[11] M. Goze, K. Yusupdjan, Nilpotent Lie Algebras. Springer, 1996.

[12] G. Grelaud, On Representations of Simply Connected Nilpotent and Solvable Lie Groups, Unpublished manuscript.

[13] D. Han and Y. Wang, Lattice Tiling and the Weyl Heisenberg Frames, Geom. Funct. Anal. 11 (2001), no. 4, 742–758.

[14] U. Hettich, R. L. Stens, Approximating a bandlimited function in terms of its samples. Approximation in mathematics (Memphis, TN, 1997). Comput. Math. Appl. 40 (2000), no. 1, 107–116.

[15] J. Lee, Introduction to smooth manifolds. Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013.

[16] V. Oussa, Computing Vergne Polarizing Subalgebras, Linear and Multilinear Algebra, Volume 63, Issue 3; (2015).

[17] V. Oussa, Sampling and Interpolation on Some Nilpotent Lie Groups, to appear in Forum Math. 2014.

[18] V. Oussa, Sinc Type Functions on a Class of Nilpotent Lie Groups, Advances in Pure and Applied Mathematics. Volume 5, Issue 1, Pages 5–19 (2014).

[19] V. Oussa, Bandlimited Spaces on Some 2-step Nilpotent Lie Groups With One Parseval Frame Generator, Rocky Mountain Journal of Mathematics, Volume 44, Number 4, 2014.

[20] I. Pesenson, Sampling of Paley-Wiener functions on stratified groups, J. Fourier Anal. Appl. 4 (1998), 271-281.

[21] G. Pfander, P. Rashkov, Y. Wang, A Geometric Construction of Tight Multivariate Gabor Frames with Compactly Supported Smooth Windows, J. Fourier Anal. Appl. 18 (2012), no. 2, 223–239. 42C15.

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